ON ARCWISE CONNECTED CONVEX SET VALUED MAPPINGS

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On Arcwise Connected Convex Set Valued Mappings

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Abstract

In this paper arcwise connected convex set valued mappings are introduced and studied. Optimality conditions involving this type of data are analyzed.

1 Introduction

In the present work we introduce a generalized definition of arcwise connected set valued functions and we study nonsmooth optimization problems involving this type of data. The present paper generalizes to set valued mapping the work by Fu and Wang [6] for vector functions. In the following $K$ will be a subset of $\mathbb{R}^n$ and $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a set valued mapping; in other words $F$ is a function from $\mathbb{R}^n$ to the power set $2^{\mathbb{R}^m}$. We will focus the attention on the problem

$$\min_{x \in K} F(x)$$

where the meaning of this is explained in the following definitions. Let $C$ be a pointed closed convex cone of $\mathbb{R}^n$.

**Definition 1.1** A point $(x_0, y_0)$ with $y_0 \in F(x_0)$ is said to be a local weak minimum point if there exists a neighbourhood $U$ of $x_0$ such that

$$F(x) \subset y_0 + (-\text{int} C)^c$$

for all $x \in U \cap K$.

**Definition 1.2** A point $(x_0, y_0)$ with $y_0 \in F(x_0)$ is said to be a local minimum point if there exists a neighbourhood $U$ of $x_0$ such that

$$F(x) \subset y_0 + (-C \setminus \{0\})^c$$

for all $x \in U \cap K$.

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We recall that the graph of \( F \) is the following subset of \( \mathbb{R}^n \times \mathbb{R}^m \):

\[
\text{graph } F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x)\}
\]

If \( F(x) \) is a closed, compact or convex we say that \( F \) is closed, compact or convex value, respectively. The following definition introduces the notion of AC set.

**Definition 1.3** [6] The subset \( K \subset \mathbb{R}^n \) is to be arcwise connected (AC set) if for any \( x, y \in K \) there exists a continuous function, called arc, \( H_{x,y} : [0, 1] \rightarrow K \) such that \( H_{x,y}(0) = x \) and \( H_{x,y}(1) = y \).

We are ready to introduce the notion of arcwise connected convex set valued mapping. This notion generalizes the notion of convex set valued mapping [8, 11].

**Definition 1.4** Let \( K \) be an AC set. A set valued mapping \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is called an arcwise connected convex cone mapping (ACCCM) if for any \( x, y \in K \) there is arc \( H_{x,y} \) such that

\[
tF(x) + (1 - t)F(y) \subset F(H_{x,y}(t)) + C
\]

for all \( t \in [0,1] \).

It is trivial to show that a convex set is an arcwise connected set and a convex set valued mapping is an arcwise connected convex cone set valued mapping.

**Definition 1.5** A given set valued mapping \( F : K \rightrightarrows \mathbb{R}^m \) is said to be convex-like (briefly CLM) if for all \( x, y \in K \) there exists a \( z \in K \) such that

\[
tF(x) + (1 - t)F(y) \subset F(z) + C.
\]

It is clear that an ACCM is also CLM.

### 2 Properties of ACCM

**Theorem 2.1** Let \( K \) be an AC set and \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) be an ACCM. If \((x_0, y_0)\) is a local minimum then it is a global minimum.

**Proof.** If \((x_0, y_0)\) is a local minimum then there exists a neighbourhood \( U \) of \( x_0 \) such that

\[
F(x) \subset y_0 + (\text{int } C)^c
\]

for all \( x \in K \cap U \). Given \( x \in K \), there exists an arc \( H_{x_0,x} \) such that

\[
tF(x) + (1 - t)F(x_0) \subset F(H_{x_0,x}(t)) + C
\]

for all \( t \in [0,1] \). If \( t \to 0 \) we have

\[
F(H_{x_0,x}(t)) + C \subset y_0 + (\text{int } C)^c.
\]
Then
\[ tF(x) + (1 - t)y_0 \subset y_0 + (-\text{int} \, C)^c \]
and this implies
\[ F(x) \subset y_0 + (-\text{int} \, C)^c \]
that is the thesis. \(\triangle\)

**Theorem 2.2** Suppose that \( F(x) \subset f(x) + C \) for all \( x \in K \) and \( f \) is an arcwise connected convex cone function. Then \( F \) is an ACCM.

**Proof.** In fact for all \( x \in K \) there exists an arc \( H_{x_0,x} \) such that
\[ tf(x) + (1 - t)f(y) - f(H_{x_0,x}(t)) \in C \]
for all \( t \in [0, 1] \). So we have
\[
F(x) + (1 - t)F(y) \subset t(f(x) + C) + (1 - t)(f(y) + C) \\
\subset tf(x) + (1 - t)f(y) + C \subset f(H_{x_0,x}(t)) + C \subset F(H_{x_0,x}(t)) + C
\]
for all \( t \in [0, 1] \). \(\triangle\)

**Theorem 2.3** Suppose that \( F(x) \subset f(x) + C \) and \( F \) is an ACCM. Then \( f \) is an arcwise connected convex cone function.

**Proof.** In fact we have
\[
F(x) + (1 - t)F(y) \subset tF(x_0) + (1 - t)F(x) \subset \\
F(H_{x_0,x}(t)) + C \subset f(H_{x_0,x}(t)) + C.
\]
\(\triangle\)

### 3 AC Generalized Derivatives

The notions of first and second order generalized derivatives we will introduce are based on the notion of local selection in the sense of the following definition.

**Definition 3.1** A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is a local selection of \( F \) at \( x_0 \) if there exists a neighbourhood \( U(f, x_0) \) such that \( f(x) \in F(x), \forall x \in U(x_0) \). We denote by \( F_{x_0} \) the set of all local selections of \( F \) at \( x_0 \).

**Definition 3.2** Let \( K \) be an AC set, \( x_0 \in K \) and \( y_0 \in F(x_0) \). Given \( x \in K \) the AC directional derivative of \( F \) with respect to \( H_{x_0,x} \) is defined as
\[
F'(x_0, y_0; H_{x_0,x}) = \left\{ t = \lim_{n \to +\infty} \frac{f(H_{x_0,x}(t_n)) - y_0}{t_n} : t_n \downarrow 0, f \in F_{x_0} \right\}
\]
Definition 3.3 Let $K$ be an AC set, $x_0, x \in K$, $y_0 \in F(x_0)$ and $z_0 \in F'(x_0; y_0; H_{x_0, x})$.

The Peano directional derivative of $F$ with respect to $H_{x_0, x}$ is defined as

$$F''(x_0, y_0, z_0; H_{x_0, x}) = \left\{ l = \lim_{n \to +\infty} 2 \frac{f(H_{x_0, x}(t_n))) - y_0 - t_n z_0}{t_n} : t_n \downarrow 0, f \in F(x_0) \right\}$$

Theorem 3.1 Let $K$ be an AC set, $x_0, x \in K$. Suppose that $(x_0, y_0)$ be a local minimum point. Then

$$F'(x_0, y_0, H_{x_0, x}) \cap -\text{int } C = \emptyset$$

for all arcs $H_{x_0, x} : [0, 1] \to K$, $H_{x_0, x}(0) = x_0$ and $H_{x_0, x}(1) = x$

Proof. Suppose that there exists an arc $H_{x_0, x}$ such that

$$F'(x_0, y_0, H_{x_0, x}) \cap -\text{int } C \neq \emptyset.$$

So there exist a selection $f \in F(x_0), t_n \downarrow 0$ and $l \in F'(x_0, y_0, H_{x_0, x}) \cap -\text{int } C$ such that

$$l = \lim_{n \to +\infty} \frac{f(H_{x_0, x}(t_n))) - y_0}{t_n}.$$

Since $(x_0, y_0)$ is a local weak minimum point we have for $n$ large enough

$$\frac{f(H_{x_0, x}(t_n))) - y_0}{t_n} \subset (-\text{int } C)^c$$

and this implies $l \in (-\text{int } C)^c$. △

Theorem 3.2 Suppose that $F$ is an ACCM at $x_0$ and $F(x) \subset f(x) + C$. Then

$$F(x) - y_0 \subset F'(x_0, f(x_0); H_{x_0, x}) + C.$$

Proof. From previous theorems we have that $f$ is an arcwise connected convex cone function, that is for all $x \in K$ there exists an arc $H_{x_0, x}$ such that

$$f(x) - f(x_0) \subset f'(x_0; H_{x_0, x}) \subset C.$$

So

$$F(x) - f(x_0) \subset f(x) - f(x_0) + C \subset f'(x_0; H_{x_0, x}) + C \subset F'(x_0, f(x_0); H_{x_0, x}) + C.$$

△
References


