STRUCTURE-BASED SVAR IDENTIFICATION

EMANUELE BACCHIOCCHI       RICCARDO “JACK” LUCCHETTI

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Emanuele Bacchiocchi* Riccardo (Jack) Lucchetti†

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Abstract

It may be desirable, for various reasons, to establish criteria for SVAR identification that do not depend on unknown parameters, but only on the set of restrictions that are imposed on the system a priori on theoretical grounds.

In the context of linear systems, this was accomplished in Johansen (1995). This paper extends and amends the approach proposed by Lucchetti (2006); we introduce a set of criteria which ensure identification independently of unknown parameters for a reasonably general class of models and discuss its possible generalization.

Keywords: SVAR, identification, Rado condition.
JEL codes: C01, C30, C32.

1 Introduction

As is well known, Structural VARs have been an essential tool for macroeconometrics for over 30 years, since they were introduced by Sims (1980). Identification issues have been a thorny issue ever since, and were first analyzed in a systematic way a decade later in the 1992 edition of Amisano and Giannini (1997).

It may be desirable, for various reasons, to establish criteria for SVAR identification that do not depend on unknown parameters, but only on the set of restrictions that are imposed on the system a priori on theoretical grounds. In other words, our focus will be on what we consider a very important issue: is it possible to assess identification of a SVAR on the basis of its constraints alone? Because if it were so, identification would cease to be a property of

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* Dipartimento di Economia, Management e Metodi Quantitativi (DEMM), Università degli Studi di Milano.
† Dipartimento di Scienze Economiche e Sociali (DISES), Facoltà di Economia "Giorgio Fuà", Università Politecnica delle Marche.
the DGP. Instead, it would be possible to consider identification an inherent characteristic of the structure that we superimpose onto the data, independent of the observable evidence. In this case, we would call the model *structurally identified*.

In the context of linear systems, this was accomplished in Johansen (1995), who mostly focused on cointegrated systems, but introduced into the economic literature a few results from matroid theory which enabled him to draw necessary and sufficient conditions for identification in linear systems in a general and elegant way.

A similar approach was used in Lucchetti (2006) to analyze identification of models, such as Structural VARs, in which linear constraints are imposed on a covariance system. As we will show, the "structure condition" defined in that article is not sufficient, as claimed. In this paper, we extend and amend the same approach and introduce a set of criteria which ensure identification independently of unknown parameters for the most widely used class of models (the C-model, in Amisano and Giannini’s taxonomy).

In doing so, we also achieve the goal of generalising the approach by Rubio-Ramirez, Waggoner, and Zha (2010), which has gained a notable popularity in recent years.

The structure of the article is as follows: in the next section, we examine the issue of identification and spell out our main result; then, in Section 3, we give a few examples of application of our criterion. Section 4 extends the results to the more general AB-model, with some examples discussed in Section 5. Section 6 concludes and outlines desirable extensions and generalizations of our present work.

## 2 Identification in the C-model

### 2.1 Basic concepts and notation

In this section, we review the identification issue in the C-model variety of Structural VARs, mainly to establish notation. Following Amisano and Giannini (1997), we call a C model a SVAR of the kind

\[
\Phi(L)y_t = \mu_t + \epsilon_t
\]

\[
\epsilon_t = Cu_t
\]

in which the \(n\)-vector of one-step-ahead prediction errors \(\epsilon_t\) is assumed to be a linear function of orthogonal structural shocks \(u_t\), so that \(V(\epsilon_t) = \Sigma = CC'\). In Amisano and Giannini’s original classification, this can be seen as a special case of the so-called AB model \(A\epsilon_t = Bu_t\) which is, however, much more seldom used in its full generality. The vast majority of empirical applications relies on a structure such as the one displayed in equation (2).

Under normality, the average log-likelihood can be written as

\[
\mathcal{L} = \text{const} - \ln|C| - 0.5 \cdot \text{tr} \left[ \hat{\Sigma}(CC')^{-1} \right]
\]

and first-order conditions for maximization are equivalent to

\[
\hat{\Sigma} = CC'
\]
where \( \Sigma \) is the sample covariance matrix of the VAR residuals; greater generality can easily be achieved by considering equation (3) simply as a set of orthogonality relationships of the kind

\[
E \left[ \varepsilon_{it} \varepsilon_{jt} - \sum_{k=1}^{n} c_{ik} c_{jk} | \mathcal{F}_{t-1} \right] = 0,
\]

which makes it possible to extend the present setup to QMLE or GMM estimation if necessary.

As is well known, this problem is underidentified, since if (3) holds, then also \( \Sigma = C \Sigma \Sigma' \) holds, where \( C' = CP \) and \( P \) is an arbitrary orthogonal matrix.

In order to achieve identification, it is necessary to make more qualifications to the above setup: the traditional solution has been to impose a system of \( p \) linear constraints on \( C \) of the kind

\[
R \text{vec } C = d. \tag{4}
\]

A special case of linear restrictions occurs when \( C \) is assumed to be lower-triangular. This was Sims’s (1980) original proposal, and is sometimes called a “recursive” identification scheme. It has a number of interesting properties, among which the fact that the ML estimator of \( C \) is just the Cholesky decomposition of \( \hat{\Sigma} \), the sample covariance matrix of VAR residuals. This has been the most frequently used variant of a SVAR model, partly for its ease of interpretation, partly for its ease of estimation.

The general case (4), first analyzed in Amisano and Giannini (1997), has been recently re-assessed and somewhat extended in a recent contribution by Rubio-Ramirez, Waggoner, and Zha (2010), who provide conditions for global identification based on linear and some nonlinear restrictions of the parameters. However, their general results are all obtained by considering equation-by-equation restrictions (which are less general than those expressible via equation 4); moreover, the rank condition they obtain rely on the availability of the unknown parameters, which is precisely what we want to avoid.

An alternative identification strategy was introduced by Rigobon (2003), who suggested to exploit heteroskedasticity to identify simultaneous equation models without having to impose restrictions on the parametric space.\(^1\) This approach has been extended to SVAR models by Lanne and Lütkepohl (2008) and generalized in Bacchiocchi (2014) and Bacchiocchi and Fanelli (2014), who combine heteroskedasticity and structural breaks with the traditional approach of imposing restrictions on the parameters of the model.

In this paper, we focus on the traditional identification strategy and we leave generalizations for future work; as a consequence of equation (4), the vector \( c = \text{vec } C \) can be written as

\[
c = \Sigma \theta + s. \tag{5}
\]

where \( \theta \) is the vector containing the \( q = n^2 - p \) free parameters (assumed variation-free). This setup is more general than it looks, since it can also handle long-run constraints of the kind proposed by Blanchard and Quah (1989)\(^2\)

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\(^1\)The idea that heteroskedasticity can be helpful in identifying econometric models has been originally proposed by Sentana and Fiorentini (2001) in a factor model context, that could be extended to SVAR models too.
or King, Plosser, Stock, and Watson (1991). However, it cannot be used with more exotic identification schemes such as inequality-based ones like in Uhlig (2005).

As argued in Amisano and Giannini (1997) and Lucchetti (2006), local identification is tied to the rank of the matrix

\[ \mathcal{H} = R (I \otimes C) \bar{D}_n \]

being full; in this case, \( \bar{D}_n \) is a \( n^2 \times \frac{n(n-1)}{2} \) matrix whose columns span the null space of the duplication matrix \( D_n \) (see Magnus, 1988). Note that, in the case of exact identification, this matrix is square; introducing extra restrictions adds extra rows, but the number of columns, which only depends on \( n \), remains unchanged.

This introduces a necessary and rather obvious order condition by which the number of linearly independent restrictions \( p \) must be at least \( \frac{n(n-1)}{2} \). However, as is well known, the order condition is necessary but not sufficient. Evidently, the elements of \( \mathcal{H} \) depend on the unknown parameters, since the \( C \) matrix is a function of \( \theta \), so it would seem that its rank cannot be established in general, but only on a case-by-case basis for each point in the parameter space \( \mathbb{R}^q \). Lucchetti (2006) introduces an additional condition, called the structure condition, which is independent of the matrix \( C \) and should ensure (together with the order condition) sufficiency for identification apart from a zero-Lebesgue-measure set in the parameter space. Unfortunately, there are cases when this does not work. In the next section, we explain why and provide a counter-example.

### 2.2 An example when the structure condition fails

As an illustrative example, consider a case with 4 variables, in which the constraints are

\[ c_{41} = c_{32} = c_{23} = c_{12} = c_{13} = c_{14} = 0 \]

Since \( n = 4 \), \( \frac{n(n-1)}{2} = 6 \) so the order condition is satisfied. In this particular case the \( \mathcal{H} \) matrix in equation 6 is square but singular for any choice of \( C \). As an example, consider this particular choice for \( C \) (non-zero entries are random uniforms, just for the sake of the example):

\[
C = \begin{bmatrix}
0.74 & 0 & 0 & 0 \\
0.40 & 0.40 & 0 & 0.13 \\
0.31 & 0 & 0.23 & 0.86 \\
0 & 0.65 & 0.25 & 0.82
\end{bmatrix}
\]

in which we take \( R \) to be

\[
R = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[2\text{The matrix } \bar{D}_n \text{ can also be defined as a basis for the space of the vectorization of } n \times n \text{ hemisymmetric matrices, so that any hemisymmetric matrix } \mathcal{H} \text{ has a vectorized form which satisfies } h = \bar{D}_n \varphi \text{ for some } n(n-1)/2 \times 1 \text{ vector } \varphi.\]
and $d = 0$. Clearly, the ordering of the rows of $R$ is immaterial for the purposes of identification. The matrix $H$, whose rank ought to be full if the model were identified, is

$$
H = \begin{bmatrix}
0.65 & 0.25 & 0.82 & 0 & 0 & 0 \\
-0.31 & 0 & 0 & 0.23 & 0.86 & 0 \\
0 & -0.40 & 0 & -0.40 & 0 & 0.13 \\
-0.74 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.74 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.74 & 0 & 0 & 0
\end{bmatrix}
$$

which is, instead, singular: the set of its three rightmost columns has evidently rank 2. Following the arguments in Lucchetti (2006), it can be shown that this implies the existence of an infinitesimal rotation matrix of the form

$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & a & b & 0 \\
0 & -a & 0 & c \\
0 & -b & -c & 0
\end{bmatrix}
$$

which, in turn, implies the existence of an orthogonal matrix which rotates, in a neighborhood of $C$, the space of structural shocks in an observationally equivalent way, so to make the system under-identified.

Furthermore, it is also instructive to explore the rank deficiency of $H$ by considering its rows as well as its columns. First, it should be noted that, under the above set of restrictions, $\Sigma_{1,4} = \Sigma_{4,1} = 0$ identically, since the first and the fourth rows of $C$ are orthogonal by construction. We conjecture that this leads to underidentification since the corresponding moment condition on $E(\epsilon_{1,1} \cdot \epsilon_{4,1})$ contains no information on any of the elements in $C$.

This is mirrored by considering that the top row in $H$ is a linear combination of the bottom three rows. In terms of identification, the meaning of this fact is that, given $\Sigma_{1,4} = 0$, the four restrictions on $c_{12}$, $c_{13}$ and $c_{14}$ could be combined into three linear restrictions on $C$. Note that this happens despite none of the linear constraints being linearly dependent on the other ones, as the constraints matrix $R$ has full row rank.

In Lucchetti (2006), it was argued that checking for the rank of $H$ is equivalent to considering the existence of solutions to the system

$$
\phi^T \bar{D}'(I \otimes R_i)[s\theta + s] = \phi^T t_i \theta + \phi^T t_i = 0 \quad \text{for } i = 1 \ldots p
$$

with

$$
T_i \equiv \bar{D}'_i(I \otimes R_i)S \\
t_i \equiv \bar{D}'_i(I \otimes R_i)s
$$

and $R_i$ is a $n \times n$ matrix whose vectorization is the $i$th row of $R$. In other words, the system is unidentified at $\theta$ if some $\phi \neq 0$ exists which satisfies the above equations for every $i$. While this is true, the check proposed in Lucchetti (2006) is flawed, and it is instructive to see why; construct the matrix

$$
T = \begin{bmatrix}
R(I \otimes S_1)\bar{D}_n \\
R(I \otimes S_2)\bar{D}_n \\
\vdots \\
R(I \otimes S_q)\bar{D}_n
\end{bmatrix}
$$
where \( S_i \) is an \( n \times n \) matrix whose vectorisation is the \( i \)-th column of \( S \); solution of the system of equations (7) amounts to finding \( \theta \) and \( \varphi \) such that

\[
(\theta' \otimes I_p) T \varphi = 0;
\]
clearly, this is always possible if the column rank of \( T \) is not full, so the structure condition in Lucchetti (2006) amounts to checking for the rank of \( T \); however, the equation above may also have solutions if a nonzero \( \varphi \) exists such that the product \( T \varphi \) is non-null, but lies in the kernel space of \((I \otimes \theta')\), that is \( Sp(\theta_\perp \otimes I_p) \).

Since the structure condition fails to capture this particular feature of the model, Lucchetti (2006)’s setup erroneously labels the above model as identified. In the next section, we propose a theorem that overcomes this deficiency, by defining a necessary and sufficient condition for structural identification.

2.3 The measure of under-identified points in the parameter space

The next result, already presented in Johansen (1995) for linear systems of equations and Rubio-Ramirez, Waggoner, and Zha (2010) for global identification in SVAR models, helps in practically checking whether a model is effectively identified or not.

**Theorem 1.** Consider the SVAR model in Eq. (2), with restrictions on the parameters as described in Eqs. (4)-(5). If the order condition is met, then either \( \mathcal{H} \) is singular for all values of \( \theta \in \mathbb{R}^q \) or the set of vectors \( \theta \) that makes it singular has zero Lebesgue measure.

**Proof.** \( \mathcal{H} \) has \( p \) rows and \( \frac{n(n-1)}{2} \) columns, so if the order condition \( p \geq \frac{n(n-1)}{2} \) is met, then the rank of \( \mathcal{H}'\mathcal{H} \) equals that of \( \mathcal{H} \). Since \( C \) in equation (6) is an affine function of \( \theta \), it follows that the determinant \( |\mathcal{H}'\mathcal{H}| \) is a polynomial of finite order in \( \theta \). As is well known (see Traynor and Caron (2005)), a finite-order polynomial function \( \mathbb{R}^q \rightarrow \mathbb{R} \) is either identically 0, or non-zero almost everywhere. Call \( \Theta_0 \) the set of \( \theta \) satisfying \( |\mathcal{H}'\mathcal{H}| = 0 \); then either \( \Theta_0 = \mathbb{R}^q \), or \( \Theta_0 \subset \mathbb{R}^q \), where \( \lambda(\Theta_0) = 0 \), and the claim follows. \( \square \)

The most powerful consequence of the above theorem is that if one \( \theta \) exists such that the model is locally identified at it, then it is locally identified almost everywhere. This result provides theoretical justification to the standard practice of checking identification by simply calculating \( \text{rk}(\mathcal{H}) \) for randomly selected values for the free parameters \( \theta \).

Moreover, it should be noted that, by the continuity of the differential function, if \( \Theta_0 \subset \mathbb{R}^q \), there exists some \( \theta^* \notin \Theta_0 \) in every neighborhood of \( \theta \in \Theta_0 \), so the subset of parameter points for which identification is attained is either empty, or dense in \( \mathbb{R}^q \).

\( ^3 \text{We use } \lambda(\cdot) \text{ for Lebesgue measure, as per standard notation.} \)
2.4 The main theorem

As we argued, a necessary and sufficient condition for the model to be structurally identified is that the matrix \( H \) in equation (6) is full rank for any possible value of the unknown parameter vector \( \theta \) (apart from a zero-measure subset). To analyze the conditions under which this is true, we study the equivalent problem of the existence of solutions to the system of equations (7).

In order to do this, define \( T(q) = \begin{bmatrix} T_1 \theta + t_1 & T_2 \theta + t_2 & \cdots & T_p \theta + t_p \end{bmatrix} \).

(8)

Note that, in the case of \( d = 0 \), all the \( t_i \)-s are zero. It is useful, at this point, to restate Rado’s (1942) theorem, in the form employed in Johansen (1995):

**Theorem 2.** Suppose you have a collection of matrices \( M_1, M_2, \ldots, M_n \); then, in order to obtain a set of linearly independent vectors \( v_1, v_2, \ldots, v_n \), where \( v_i = M_i \Lambda_i \), a necessary and sufficient condition is that for all \( k = 1, \ldots, n \) and all sets of indices \( 1 \leq i_1 < \cdots < i_k \leq n \),

\[
\text{rk } [M_{i_1} \ldots | M_{i_k}] \geq k.
\]

Now, define \( T_i = (T_i \mid t_i) \) and consider

\[
T = \begin{bmatrix} T_1 & T_2 & \cdots & T_p \end{bmatrix},
\]

whose size is \( \frac{n(n-1)}{2} \times pq \) with \( pq > \frac{n(n-1)}{2} \). This matrix collects all the vectors composing \( T_i, i = 1 \ldots p \). The next theorem, based on the result in Theorem 1, when combined with the order condition \( p \geq \frac{n(n-1)}{2} \), provides a necessary and sufficient condition for the \( T(q) \) matrix to have full rank \( \frac{n(n-1)}{2} \), independently of the free parameters \( \theta \).

Put a different way, the theorem provides a necessary and sufficient condition for the model to be identified that is only based on the imposed restrictions, and that can be easily checked before the estimation process. As in Johansen (1995), it can be said that identification of the SVAR is a characteristic of the model, rather than being tied to the parameter matrix \( C \).

**Theorem 3.** Consider the specification in equations (2)-(3) and the set of restrictions given by equations (4)-(5), with \( p \geq \frac{n(n-1)}{2} \). Given an integer \( 1 \leq k \leq \frac{n(n-1)}{2} \) and a corresponding integer \( h = p - \frac{n(n-1)}{2} + k \), consider all possible sets of \( h \) increasing integers between 1 and \( p \): \( 1 \leq i_1 < i_2 < \cdots < i_h \leq p \); if

\[
\text{rk } (T_{i_1} | \ldots | T_{i_h}) \geq k,
\]

for any \( k \), then the function \( \text{rk}[T(\theta)] \) equals \( \frac{n(n-1)}{2} \) almost everywhere in \( \mathbb{R}^\theta \).

We call the Rado condition the condition by which the inequalities (9) are simultaneously satisfied. Therefore, the SVAR model is structurally identified if and only if the Rado condition is met, except for a zero Lebesgue measure set of values for \( \theta \).

\[4\]This, for example, is the case of recursive identification and is by far the most common setup in the C model.
Proof. The theorem directly derives from Theorem 1 and Theorem 2. Proving the necessity condition is trivial: if the model is identified, then $T(\theta)$ is of full column rank. As a consequence, it will be possible to choose some vectors $\theta_i$, for $i = 1 \ldots p$, or equivalently (almost surely) a common vector $\theta$, such that the $n(n-1)/2$ columns in $T(\theta)$ are linearly independent. This implies that, for each $k$, the matrix $(T_i \mid \ldots | T_k)$ must contain at least $k$ linearly independent vectors. If, instead, the conditions for theorem 2 are not met for a generic set of vectors $\theta_i$, then, given Theorem 1, they will not be met anywhere in the parametric space, so that $\Theta_0 = \mathbb{R}^l$.

As for sufficiency, consider the just identified case first, in which $p = \frac{n(n-1)}{2}$. If the conditions of Theorem 2 are met, then it is possible to choose vectors $T_i\theta_i + t_i$ that are linearly independent such that the matrix

$$[T_1\theta_1 + t_1 \mid T_2\theta_2 + t_2 \mid \ldots \mid T_p\theta_p + t_p]$$

is invertible for such specific $\theta_i$ vectors. But if this is possible, then the possibility that $\Theta_0 = \mathbb{R}^l$ is ruled out; therefore, even if a specific point $\theta$ may belong to the zero-measure set $\Theta_0$, identification is attained at some other (arbitrarily close) point $\theta^*$.

Finally, consider the over-identified case, i.e. $p > \frac{n(n-1)}{2}$. In this case, the number of columns in $T(\theta)$ as defined in Eq. (8) is larger than the number of rows. Thus, it is impossible to check a condition for all columns of the $T(\theta)$ to be linearly independent, as at least $p - \frac{n(n-1)}{2}$ will be obtained as linear combination of the others. Therefore, we have to verify that the spaces spanned by $(T_1 \mid \ldots | T_p)$ will generate exactly $n(n-1)/2$ linearly independent vectors. To prove the result it is sufficient to start our rank checks at $h = \left[p - \frac{n(n-1)}{2} + 1 \right]$ instead of at $h = 1$.

It is perhaps useful to give an explicit example of the condition on Theorem 3: Consider a trivariate system with 4 restrictions so that $n = 3$, $\frac{n(n-1)}{2} = 3$ and $p = 4$. Theorem 3 states that we first have to check the rank of all the matrices of the type $(T_{i_1}|T_{i_2})$, that is for $(i_1, i_2)$ equal to

$$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4);$$

the rank of all the above combinations should be at least 1. Then, we consider matrices of the kind $(T_{i_1}|T_{i_2}|T_{i_3})$, so we need the combinations below:

$$(1,2,3), (1,3,4), (2,3,4);$$

all these matrices must have rank at least 2. Finally, we check for the only combination of 4 submatrices: $(T_1|T_2|T_3|T_4)$; this matrix must have rank 3.

If $p \leq \frac{n(n-1)}{2}$ the order condition fails, and the model cannot be identifiable as there are not enough empirical moments in the covariance matrix $\Sigma$ to estimate the free parameters of the model. If, instead, $p > \frac{n(n-1)}{2}$, the number of imposed restrictions is large enough compared to the number of distinct elements in $\Sigma$, and the necessary condition is met. At this stage, the condition in Eq. (9) becomes a necessary and sufficient condition for the identifiability of the SVAR model.
3 A few example C-models

The next examples provide some details on the implementation of the identification condition developed in Theorem 3 presented in Section 2.

3.1 Unidentified bivariate SVAR

This example is analyzed in detail both in Amisano and Giannini (1997) and in Lucchetti (2006). Consider the following bivariate SVAR model where the matrix of contemporaneous relations is defined as

$$C = \begin{pmatrix} \theta_1 & \theta_2 \\ -\theta_2 & \theta_1 \end{pmatrix}.$$

The structure of the $C$ matrix presents a problem similar to the one discussed in Section 2.2, i.e. it implicitly imposes a covariance matrix of the reduced form as follows

$$CC^\prime = \begin{pmatrix} \theta_1^2 + \theta_2^2 & 0 \\ 0 & \theta_1^2 + \theta_2^2 \end{pmatrix},$$

where the covariance or the residuals is restricted to be zero and the two variances are equal. The model thus presents two parameters to be estimated but only one empirical moment. The matrices of restrictions are

$$R = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{D}_n = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

The $T_1$ and $T_2$ matrices become

$$T_1 = D_2^\prime (I_2 \otimes R_1^\prime) S =$$

$$= \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

$$T_2 = D_2^\prime (I_2 \otimes R_2^\prime) S =$$

$$= \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

The order condition is clearly met ($n(n-1) = 1$ and $p = 2$). Using Theorem 3, however, gives the following rank conditions

$$\text{rk} (T_1) = 0 < 1, \quad \text{rk} (T_2) = 0 < 1, \quad \text{rk} (T_1 \mid T_2) = 0 < 2.$$

The model, thus, cannot be identifiable.
3.2 Trivariate Cholesky SVAR

Consider the triangular trivariate SVAR model. Rubio-Ramirez, Waggoner, and Zha (2010) prove that this model is not simply locally identified, but also globally. The matrix of simultaneous relations is defined as

$$ C = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} $$

where asterisks (‘*’) denote unrestricted coefficients. The matrices of restrictions can be written as

$$ R = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. $$

The order condition is clearly satisfied. The $T_1$, $T_2$ and $T_3$ matrices are

$$ T_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. $$

The rank condition provided by Theorem 3 gives the following results

$$ \text{rk} (T_1) = 1 \geq 1, \quad \text{rk} (T_2) = 1 \geq 1, \quad \text{rk} (T_3) = 2 \geq 1 $$

$$ \text{rk} (T_1 | T_2) = 2 \geq 2, \quad \text{rk} (T_1 | T_3) = 2 \geq 2, \quad \text{rk} (T_2 | T_3) = 2 \geq 2 $$

$$ \text{rk} (T_1 \mid T_2 \mid T_3) = 3 \geq 3 $$

indicating that the model is identifiable.

3.3 Unidentified four-variable SVAR

In this section we use the new results in Theorem 3 to show that the unidentified SVAR model described in Section 2.2 can now be correctly detected. The necessary and sufficient condition for identification in Eq. (9) suggests to calculate the rank of all possible combinations of the $T_1, \ldots, T_6$ matrices, where
\( p = 6 \) indicates the number of restrictions (rows of \( R \)). This is rather boring to do by hand, but is also easy to check automatically, via appropriate software. Considering the example described in Section 2.2, the rank condition is verified for singles, pairs and triples of \( T_i, i = 1, \ldots, 6 \), but fails when we check

\[
\text{rk} \left( T_1 \mid T_4 \mid T_5 \mid T_6 \right) = 3 \leq 4
\]

indicating that the model is under-identified, in accordance to the previous analysis reported in Section 2.2.

### 4 Identification in the AB-model

Following the terminology of Amisano and Giannini (1997) and Lütkepohl (2006), the AB-model considers possible simultaneous relationships among the observable variables and the structural shocks. Classic empirical applications using this specification can be found, among many others, in Bernanke (1986), Blanchard (1989) and Blanchard and Perotti (2002). Concentrating out the dynamic part of the VAR, the simultaneous relations are defined by

\[
A \varepsilon_t = B \eta_t
\]

where, as before, \( \varepsilon_t \) is the one-step-ahead prediction errors, with \( E(\varepsilon_t\varepsilon_t') = \Sigma \), and \( \eta_t \) collects the standardized and uncorrelated structural shocks. The specification in Eq. (10) naturally induces a set of non-linear restrictions on the parameter space given by

\[
A \Sigma A' = BB'.
\]

The number of parameters, here, is clearly \( 2n^2 \), while that of empirical moments remains ‘at most’ \( n(n + 1)/2 \), i.e. the number of elements in the estimable covariance matrix of the residuals \( \Sigma \). As a consequence, at least \( p = 2n^2 - n(n + 1)/2 = n^2 + n(n - 1)/2 \) restrictions on the parameters must be included for identification.

The first systematic contribution in terms of the identification has been provided by Amisano and Giannini (1997). Write the \( p_a \) and \( p_b \) independent restrictions on \( A \) and \( B \) as

\[
R_{a}\text{vec } A = r_{a} \quad \text{and} \quad R_{b}\text{vec } B = r_{b},
\]

or equivalently, in explicit form,

\[
\text{vec } A = S_{a}\theta_{a} + s_{a} \quad \text{and} \quad \text{vec } B = S_{b}\theta_{b} + s_{b};
\]

where \( \theta_{a} \) and \( \theta_{b} \) are of dimension \( q_{a} \times 1 \) and \( q_{b} \times 1 \), respectively, and contain the \( q = q_{a} + q_{b} \) free parameters to estimate; Amisano and Giannini (1997) introduce a necessary and sufficient identification condition based on the following \( p_{b} \times [n(n - 1)/2 + q_{a}] \) matrix

\[
R_{b} \left( B \otimes BB' \right) \left[ \tilde{D}_{a} \right] - \left[ A^{-1} \otimes (BB')^{-1} \right] S_{a}
\]

that must have full column rank for the model to be identified. As for the C-model previously discussed, this matrix depends on the true values of \( A \) and \( B \), that are unknown.
In what follows, we use the intuition in Lucchetti (2006) and check for the existence of a pair of matrices

\[ A_s = A + dA = (I + Q) A \]
\[ B_s = B + dB = (I + Q) B (I + H) \]

that are observationally equivalent to \( A \) and \( B \) in a neighborhood of the two true matrices \( A \) and \( B \).

The two infinitesimal transformation matrices \( (I + Q) \) and \( (I + H) \) must obey different requisites: \( (I + H) \) should be orthogonal, while invertibility is sufficient for \( (I + Q) \). This implies that, while it is sufficient to consider a simple infinitesimal non-zero matrix \( Q \neq 0 \), \( H \) must be an infinitesimal rotation with \( H = -H' \). The two new matrices of parameters \( A_s \) and \( B_s \) will be admissible if

\[
R_a da = R_a (I \otimes Q) (S_a \theta_a + s_a) = 0 \\
R_b db = R_b [(I \otimes Q) + (H' \otimes I)] (S_b \theta_b + s_b) = 0.
\]

where \( a = \text{vec} A \) and \( b = \text{vec} B \). With a little algebra, the identification condition can be shown to be equivalent to the following system of equations\(^5\)

\[ [q' \mid \phi'] V(\theta) = 0 \]  \hspace{1cm} (15)

where

\[
V(\theta) = \begin{bmatrix}
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
T_1^a \theta + f_1^a & \cdots & T_1^b \theta + f_1^b & \cdots & T_p^a \theta + f_p^a & \cdots & T_p^b \theta + f_p^b
\end{bmatrix}
\]  \hspace{1cm} (16)

with

\[
T_i^a = D_n^a (I_n \otimes R_{b,j}^i) S_a \\
T_i^b = D_n^b (I_n \otimes R_{b,j}^i) S_b
\]

in which \( R_{a,i} \) and \( R_{b,j} \), similarly to \( R_i \) defined in Section 2.1, are defined such that their vectorization is equal to the \( i \)-th row of \( A \) and \( B \), respectively. The matrix \( K_{nn} \) is the \( n^2 \times n^2 \) matrix defined by the property \( K_{nn} \text{vec} (A) = \text{vec} (A') \).\(^6\)

If the system admits as its unique solution the null vector \([q' \mid \phi'] = [0] \), or equivalently, \( V(\theta) \) has full row rank, then the model is identified. Obviously, \( V(\theta) \) is a function of the free parameters \( \theta \), but then we can employ Theorem 2 and obtain a necessary and sufficient condition independent of \( \theta \) like we did for the C-model, discussed in Theorem 3.

For convenience, define

\[
V(\theta) = [V_1 \mid \cdots \mid V_{p_a} \mid V_{p_a+1} \mid \cdots \mid V_{p_a+p_b}] (I_p \otimes \theta) = V (I_p \otimes \theta)
\]

\(^5\) See Lucchetti (2006), pag 247, Eq. (23).

\(^6\) It can be shown that \( K_{nn} \) is symmetric and orthogonal; see for instance Magnus and Neudecker (1988, p. 46).
where $\theta = (\theta_p' \mid 1 \mid \theta_q' \mid 1)'$, and

$$V_i = \begin{cases} 
\begin{pmatrix} 
U^a_i & u^a_i & : & 0 \\
0 & : & 0 
\end{pmatrix} & \text{for } 1 \leq i \leq p_a \\
\begin{pmatrix} 
U^b_i & u^b_i & : & 0 \\
0 & : & T^b_i & t^b_i 
\end{pmatrix} & \text{for } p_a + 1 \leq i \leq p_a + p_b 
\end{cases}$$

The theorem can now be stated as:

**Theorem 4.** Consider the specification in Eqs. (10)-(11) and the set of $p$ restrictions given by Eqs. (12)-(13), with $p = p_a + p_b \geq n^2 + \frac{n(n-1)}{2}$. Given an integer $1 \leq k \leq \left( n^2 + \frac{n(n-1)}{2} \right)$ and a corresponding integer $h = p - \left( n^2 + \frac{n(n-1)}{2} \right) + k$, consider all possible sets of $h$ increasing integers between 1 and $p$: $1 \leq i_1 < \ldots < i_h \leq p$, if

$$\text{rk } (V_{i_1} | \ldots | V_{i_h}) \geq k,$$

(17)

for any $k$, then the function $\text{rk}[V(\theta)]$ equals $n^2 + \frac{n(n-1)}{2}$ almost everywhere in $\mathbb{R}^q$, where $q = 2n^2 - p$ is the number of free parameters.

We call the Rado condition the condition by which the inequalities (17) are simultaneously satisfied. Therefore, the SVAR model is structurally identified if and only if the Rado condition is met, except for a zero Lebesgue measure set of values for $\theta$.

**Proof.** The proof of this theorem follows the same reasoning of the proof of Theorem 3. Here, the $V_i$ matrices substitute the $T_i$ matrices, while the number of rows of $V$ is $n^2 + \frac{n(n-1)}{2}$ rather than $\frac{n(n-1)}{2}$. When the number of restrictions is strictly equal to what required by the necessary condition for identification, then the set of indices for which the rank is checked in Eq. (17), goes from 1 to $p$. If, instead, the number of restrictions exceeds the minimum required (the model is potentially overidentified) the rank in Eq. (17) is checked for all indices from $p - \left( n^2 + \frac{n(n-1)}{2} \right) + 1$ to $p$, where $p - \left( n^2 + \frac{n(n-1)}{2} \right)$ is the degree of overidentification. The reason is that, when the model is potentially overidentified, the number of columns of $V(\theta)$, that must be of full-row rank, is larger than the number of rows. What is important, however, is that there must be $n^2 + \frac{n(n-1)}{2}$ linearly independent column among the $p > n^2 + \frac{n(n-1)}{2}$ columns composing the $V(\theta)$ matrix.

5 AB-models: two examples

In this section we present two examples on the implementation of the previously developed strategy to study identification in the general AB-model. The former is an artificial bivariate model of some theoretical interest; the latter is a trivariate SVAR model presented by Blanchard and Perotti (2002) in a well-known empirical application.
5.1 Bivariate AB-model

Consider the bivariate AB-SVAR model in Eq. (10) with
\[
A = \begin{pmatrix} 1 & \theta_1 \\ -\theta_1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \theta_2 & 0 \\ 0 & \theta_3 \end{pmatrix}.
\] (18)

The non-linear restrictions implicitly imposed are given by
\[
\Sigma = A^{-1}BB'(A')^{-1} = \frac{1}{(1+\theta_1^2)^2} \begin{pmatrix} \theta_2^2 + \theta_1^2\theta_3^2 & -\theta_1\theta_2^2 + \theta_1\theta_3^2 \\ -\theta_1\theta_2^2 + \theta_1\theta_3^2 & \theta_3^2 + \theta_1\theta_2^2 \end{pmatrix}.
\]

Global identification dictates that a unique solution exists to the non-linear system of three equations in three unknowns connecting the empirical moments in \(\Sigma\) to the parameters \(\theta_1, \theta_2\) and \(\theta_3\). It is interesting to note that the Rubio-Ramirez, Waggoner, and Zha (2010) approach for checking for global identification cannot be applied to AB-SVAR models. It is true that this model could be turned into a C-model, but then identification would rely on a non-linear, cross-equation constraint, which is also impossible to handle in Rubio-Ramirez, Waggoner, and Zha’s approach.

Local identification, instead, can be checked by using the result in Theorem 4, when considering the set of restrictions given by
\[
R_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad r_a = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad R_b = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad r_a = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]
\[
S_a = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad s_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad S_b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad s_a = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Checking for the rank of all combinations of indices as discussed in Theorem 4 shows that the model is locally identified. However, if one imposes the further restriction that \(\theta_2 = \theta_3\), then the relation between the empirical moments and the parameters becomes
\[
\Sigma = \begin{pmatrix} \theta_2^2 + \theta_1^2\theta_3^2 & 0 \\ 0 & \theta_3^2 + \theta_1\theta_2^2 \end{pmatrix}
\]
that is similar to the C-model investigated in Section 3.1. As expected, although the number of restrictions is larger than required, the Rado condition fails and the model is not identified.

5.2 Blanchard-Perotti AB-model

The following model is inspired by the seminal contribution by Blanchard and Perotti (2002) on the effects of fiscal shocks to the real economy. They consider
a three-dimensional VAR for taxes, spending and GDP, all in real per capita terms. Once concentrating out the reduced-form parameters, the model can be written as

\[
\begin{pmatrix}
1 & 0 & -a_1 \\
0 & 1 & -b_1 \\
-c_1 & -c_2 & 1
\end{pmatrix}
\begin{pmatrix}
l_t \\
g_t \\
x_t
\end{pmatrix}
= \begin{pmatrix}
a_3 & a_2 & 0 \\
b_2 & b_3 & 0 \\
0 & 0 & c_3
\end{pmatrix}
\begin{pmatrix}
\epsilon_t^l \\
\epsilon_t^g \\
\epsilon_t^x
\end{pmatrix}
\]

(19)

where \(\epsilon_t = (\epsilon_t^l, \epsilon_t^g, \epsilon_t^x)'\) is the vector of reduced-form residuals while \(u_t = (u_t^l, u_t^g, u_t^x)'\) is the vector of uncorrelated unit-variance structural shocks.

The model is clearly not identified as the number of parameters to estimate exceeds the number of distinct elements in the reduced-form covariance matrix \(9 > \frac{n(n-1)}{2} = 6\). For the order condition to hold, then, at least three more restrictions must be imposed.

Starting from the specification in Eq. (19), here below we consider alternative combinations of further restrictions (more or less backed by economic intuition) and, for each of these, check for the identification of the model using the Rado condition developed in Theorem 4. All the results are confirmed by the traditional Amisano and Giannini’s approach by calculating the rank of the matrix in Eq. (14) for randomly generated A and B. A horizontal bar over a parameter, as in \(a_1\), means that the parameter is calibrated to some non-zero value, usually on the basis of previous studies.

5.2.1 Further restrictions I

\(b_1 = 0\quad a_1 = \bar{a}_1\quad b_2 = 0\)

or alternatively

\(b_1 = 0\quad a_1 = \bar{a}_1\quad a_2 = 0\)

This is the specification used by the authors in their empirical application. In particular, they justify the choice of \(b_1 = 0\) by saying that it is not possible to identify any automatic feedback from economic activity to government spending. Moreover, the elasticity to output of net taxes (tax minus transfer) can be calibrated by using external information and thus considered as fixed within the SVAR framework. Concerning the last restriction, the authors state that there are no convincing ways to establish \textit{a priori}, after a simultaneous and unexpected change in taxes and expenditure, which of the two variables, taxes and public spending, reacts first. For this reason they present both the results with \(b_2 = 0\) or alternatively \(a_2 = 0\).

These three further restrictions satisfy the necessary order condition of identification, being \(p = p_a + p_b = 7 + 5 = 12\) equal to \(n^2 + \frac{n(n-1)}{2} = 9 + 3 = 12\). Moreover, using the Rado condition defined in Eq. (17), it is possible to show that, independently of the free parameters, the model is almost everywhere identified. In fact, given the restrictions \(b_2 = 0\) and \(a_1 = \bar{a}_1\), the \textit{cyclically adjusted} tax and public spending, \(\hat{l}_t = l_t - \hat{a}_1 x_t\) and \(\hat{g}_t = g_t - b_1 x_t\) respectively, can be used as instruments to estimate \(c_1\) and \(c_2\) in a regression of \(x_t\) on \(l_t\) and \(g_t\). Furthermore, the consistently estimated \(\hat{l}_t\) and \(\hat{g}_t\) allow to estimate the remaining two parameters \(a_2\) and \(b_1\), provided that, in turn, one of the two is
restricted to zero. Finally, the diagonal elements of the B matrix are the standard deviations of the estimated structural shocks $u_t = (u^g_t, u^\pi_t, u^\sigma_t)^T$.

5.2.2 Further restrictions II

$$b_1 = 0 \quad a_1 = \bar{a}_1 \quad c_1 = 0$$

or alternatively

$$b_1 = 0 \quad a_1 = \bar{a}_1 \quad c_2 = 0$$

This set of restrictions, differently from the previous one, imposes that one of the two elasticities of output to taxes and public spending is null, i.e. that output does not respond, within the quarter, to changes in the level of taxes or spending. Moreover, we consider that both $g_t$ and $l_t$ react to unexpected shocks in taxes and spending $u^g_t$ and $u^\sigma_t$, respectively.

As before, the number of restrictions satisfies the necessary order condition. Looking at the sufficiency, instead, the Rado condition is now not supported by the set of restrictions, indicating that the model is not identified. In fact, although $b_1 = 0$ and $a_1 = \hat{a}_1$ help consistently estimating $c_1$ and $c_2$, this does not allow to consistently estimate $a_2$ and $b_2$.

5.2.3 Further restrictions III

$$b_1 = 0 \quad a_1 = \bar{a}_1 \quad a_2 = b_3$$

In this case, the further restrictions, other than, as before $b_1 = 0$ and $a_1 = \hat{a}_1$, impose that $g_t$ and $l_t$ react to $u^\pi_t$ through coefficients of the same magnitude. This would be a very interesting case to analyse from an economic point of view because it would describe an economy that, for some reason, follows a policy of automatic stabilizing certain budget items. For example, during the recent financial crisis, the Italian budget contained certain tax increases (mostly excises), that were supposed to be applied automatically and conditionally to a certain deficit target not being met. This was mainly conceived as a way to reassure the markets about the commitment by the Italian government to fiscal discipline, and was repeatedly interpreted as such by Italian and EU authorities in public statements.

From a strictly econometric point of view, this restriction is of particular interest because it involves coefficients in two different equations; this type of constraint cannot be handled in the framework proposed by Rubio-Ramirez, Waggoner, and Zha (2010), in which cross-equation restrictions are not considered. And yet, in this scenario, a cross-equation restriction such as $a_2 = b_3$ would be very a natural and direct way to translate an institutional feature into a parametric restriction.

The necessary order condition continues to hold, but differently from the previous case, now the Rado condition holds too, so the model is locally identified. This last cross-equation restriction, apart from being interesting from an economic point of view is in fact indispensable for identifying all the parameters of the model.
6 Conclusions and future directions

In this paper, we have developed a method for checking whether a given SVAR is identified on the pure basis of the restrictions the covariance matrix is subject to, independently of its parameters. Therefore, it is possible to investigate identification prior to estimation. Our analysis is not limited to C-models, but also extends to the more general AB model.

A very interesting research avenue lies in investigating qualitatively the possible under-identification issues that derive from the relationship between observables and parameters; in the example we gave in subsection 2.2, lack of identification was directly linked to the absence of information contained in one of the moment conditions (3). It may be possible to derive a formal criterion for detecting such cases by considering which one, of the set of inequalities (9), fails to hold. The example given in subsection 3.3 looks quite promising in that respect.

A more difficult open question is whether our theorem can be combined with extra conditions for checking global identification, rather than local. This will be the object of future work, as well as the usage of the results presented here for dealing with the heteroskedastic case and make the analysis in Bacchiocchi (2014) more general and rigorous.

References


