PUBLIC CAPITAL, PRIVATE CAPITAL, AND ECONOMIC GROWTH

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ABSTRACT

An endogenous growth model is presented in which productive government expenditure takes the form of a stock. Private and public capital interact with each other in two different ways. The first takes place in the final output sector and depends on the specification of the aggregate production function (Cobb-Douglas vs. CES). The second has to do with the rates of investment in the two types of capital and arises from the law of motion of public capital. The share of public capital devoted to output production can be exogenous or endogenous. Our results suggest that when this share is exogenous along the balanced growth path the optimal growth rate of the economy depends positively on the degree of complementarity between the investments in the two kinds of capital, irrespective of the form of the aggregate production function. This is also true when the share of public capital devoted to output production is endogenous, as long as the inverse of the intertemporal elasticity of substitution in consumption is sufficiently large. When the technology for final output production is CES and the social planner can choose the fraction of public capital to be devoted to goods-production, optimal growth crucially depends on the elasticity of substitution between the two forms of capital in the production of goods. We analyze the conditions for an increase in this elasticity to yield either a positive, or a negative, or else an ambiguous effect on the economy’s optimal growth rate. Unlike Barro (1990), the relationship between optimal growth and the share of productive government expenditure in GDP is nonlinear and characterized by threshold-effects.

KEY WORDS: Economic Growth; Complementarity/Substitutability; Public Capital; Private Capital

JEL CODES: O41; E60; H54

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1. Introduction

The analysis of the effects of public spending on economic growth is still at the core of economists’ research agenda, both from a theoretical and empirical perspective. Since the seminal contribution of Barro (1990), endogenous growth theory has included public services into firms’ production function or consumers’ utility showing that government activity is able to produce significant, positive and permanent effects on long-run economic growth. At the same time, many empirical studies have analyzed the link between the various components of government spending and economic growth highlighting the importance of the functional classification of public expenditure for long-run growth.\(^1\)

The present paper has three main objectives. The first consists in extending Barro (1990)’s model in order to re-examine the role of productive government activity in long-run optimal growth. Unlike Barro (1990), where it is a flow-variable, we model public expenditure as a stock and study the effect (if any) of a change in the degree of complementarity/substitutability between private and public capital investments on the optimal growth rate of the economy. Using an endogenous growth model in which both types of capital enter the aggregate production function of final output as inputs, the second aim of this paper is to investigate under which conditions an increase in the extent at which the two forms of capital are substitutes/complementary for each other in the production of goods can yield a positive impact on economic growth. Finally, we use our setting to analyze whether and, eventually, how results concerning the link between optimal growth and the GDP-share of public capital may change in comparison with a model economy in which there exists only one stock-variable (private capital) and the current flow of productive public expenditure is a time-invariant fraction of current output (Barro, 1990).

In the paper we interpret public capital as (public) infrastructural capital. Our interest in this interpretation stems from the crucial role played by transportation, energy and telecommunication networks in production and economic growth. As surveyed by Straub (2008), there are several channels through which public infrastructures may contribute to long-run growth. Apart from the traditional productivity-effect, which hinges upon the complementarity between, say, an efficient public transportation system and the area’s private investments (i.e., the complementarity between public and private capital in goods production), several other indirect effects may arise, including the reduction of private investment costs, an increase in labor efficiency, a positive impact on human capital through increases in people’s health and education levels, and possible economies of scale and scope. In a recent contribution, Duggal et al. (2007), using US data, provide clear evidence that public infrastructures, along with private ICT-capital, explain most of economic growth in growth-accounting exercises.\(^2\)

The first to specify productive government expenditure as a stock were Futagami et al. (1993).\(^3\) They introduce government capital, together with private capital, in an endogenous growth model with two stock-variables and show how, contrary to models in which government expenditure is taken as a flow, this can produce transitional dynamics. Moreover, the authors use their model to study the dynamic effects of a change in the income tax rate on the transitional path of the economy and on lifetime welfare. However, they do not address the issue of how a change in the degree of complementarity/substitutability between public and private capital investments and between public and private capital stocks in goods production can ultimately affect the long-run optimal growth rate of an economy along the balanced growth path (henceforth BGP).

The main novelty of our contribution consists in assuming that productive public capital (a reproducible input along with private capital) is employed in part in the production of final goods (consumption goods, or final output) and in part in the process of accumulation of new public

\(^1\) Results in this research area are, however, mixed. As examples see, among others, Ghosh and Gregoriu (2008) and Devarajan et al. (1996).

\(^2\) For a recent review of the empirical literature on the relationships between public capital and economic growth, with an explicit focus on public infrastructure-capital, see Romp and de Haan (2007).

\(^3\) See also Baxter and King (1993).
capital. Through this modeling strategy we are able to take into account the existence of two possible sources of interaction between the two types of capital–stocks. First of all, they are inputs (with different degrees of complementarity/substitutability, depending on the form of the aggregate technology) in the production of final output; secondly, the two types of investment can be either complementary or substitutes for each other, depending on the sign of a key technological parameter. To be more concrete, we start by considering an aggregate Cobb-Douglas production function with constant returns to scale to private and public capital together. The share of public capital that is used in current production, which hereafter we denote by $s_Y$, can be thought of as exogenous or endogenous. In the first case, we are describing a *constrained* economy in which the allocation of public capital to final output production is set independently of the maximization of any objective function. On the other hand, if the social planner can determine $s_Y$ as the solution to an intertemporal optimization problem, we are in an *unconstrained* economy. In order to take explicitly into account the different degree of complementarity/substitutability between private and public capital stocks in final output production and the way this variable may affect optimal growth, in the second part of the paper we use a more general Constant Elasticity of Substitution (CES) aggregate technology. Even under the CES–model we present separately the results for the two cases in which $s_Y$ is exogenous and endogenous, respectively.

The results of the model provide new insights on the role of public infrastructure capital in economic growth and have interesting policy implications as far as the issue of the relationship between optimal growth and the share of public capital in GDP is concerned. These results can be summarized as follows. In the benchmark model (Cobb-Douglas production function and exogenous allocation of public capital to final output), a higher degree of complementarity between investments in private and public capital leads unambiguously to higher growth. Moreover, along the BGP the optimal growth rate of the economy is a negative function of the (exogenous) share of government capital devoted to goods production. Moving towards an identical model in which the allocation of public capital to final output production is fully endogenous rather than exogenous, we see that the long-run optimal balanced growth rate of the economy is still positively affected by an increase of the degree of complementarity between private and public capital investments, provided that the inverse of the intertemporal elasticity of substitution in consumption is sufficiently large. We show that this condition is easily checked with logarithmic instantaneous preferences. We also find that another crucial determinant of the economy’s growth rate is the share of public capital in GDP. At this regard, we provide simple conditions under which, unlike Barro (1990) where it is always decreasing, the relationship between this share and optimal growth can be positive. When the technology for final output production becomes CES and the social planner takes the allocation of public capital to goods production as exogenously given, the common optimal balanced growth rate of the economy is the same as in the respective Cobb-Douglas case. With endogenous allocation of public capital, instead, we observe that optimal growth is crucially dependent upon the parameter determining the elasticity of substitution between the two forms of capital in the consumption goods industry. A priori, the relationship between economic growth and this parameter is not that clear. However, we study the conditions for these two variables to be either always positively, or always negatively, or else ambiguously related to each other. Unlike the exogenous-case, when the allocation of public capital to goods production is endogenous a closed-form solution for the economy’s optimal balanced growth rate cannot in general be found, either with Cobb-Douglas or CES aggregate production function. However, we analyze the constraints on the parameters of the model that need to be met for a positive and constant optimal growth rate of the economy to exist and to be unique in both cases.

The paper is organized as follows. In Section 2 we present and discuss the main characteristics and hypothesis of our theoretical framework. Sections 3 and 4 present the two formal models with Cobb-Douglas aggregate production function in the goods sector. In Section 3 the share of public capital used in production is exogenous, whereas it is endogenous in Section 4. Section 5
generalizes the previous two models by employing a CES aggregate technology. In Section 6 we study the relationship between the share of public capital in GDP and optimal growth within the model featuring Cobb-Douglas production function and endogenous allocation of public capital across sectors and compare our results with those obtained in the seminal Barro (1990)’s paper. The last section summarizes and concludes.

2. The Economic Environment

Consider an economy populated by many structurally identical, infinitely lived households whose number is constant and normalized to unity. Each member of the representative dynastic family works and supplies inelastically one unit of labor-services per unit of time. So, there is no endogenous labor/leisure choice and population (namely, the size of the representative household) corresponds to the number of workers/consumers. Population is constant and the economy is closed. A homogeneous final good, \( Y \) (the numeraire-good in the model) is produced through an aggregate Cobb-Douglas production function. Since there is a large number of atomistic firms producing this good, we analyze the choices of a representative firm with production technology given by:

\[
Y_t = AK_t^{\alpha} L_t G_t^{\alpha}, \quad \alpha \in (0,1).
\]  

(1)

In Eq. (1) \( Y \) denotes aggregate GDP, \( A > 0 \) is a time-invariant parameter representing total factor productivity, \( L \) is the constant labor–force (population), \( K \) is the stock of private physical capital and \( G_t \) stands for the stock of productive public (infrastructure) capital employed in goods-production. Finally, \( \alpha \) and \((1 - \alpha)\) are, respectively, the factor–shares of public and private capital in GDP. The main reason for considering \( A \) and \( L \) as given constants is that we are not interested in modeling technological progress and the dynamics of population. At this aim, we further simplify the analysis and normalize these two variables to one (we set \( A = L = 1 \) for each \( t \geq 0 \)). Hence, the aggregate production function reads as:

\[
Y_t = K_t^{\alpha} G_t^{\alpha}, \quad \alpha \in (0,1).
\]  

(1’)

With \( A = 1 \) this Cobb–Douglas formulation coincides with that used by Barro (1990, p. S107, Eq. 10). Using Barro (1990, p. S106)’s words: “...I assume that (public) services are provided without user charges and are not subject to congestion effects (which might arise for highways or some other public services). That is the model abstracts from externalities associated with the use of public services. I consider initially the role of public services as an input to private production. It is this productive role that creates a potentially positive linkage between government and growth... That is, production involves decreasing returns to private inputs if the (complementary) government inputs do not expand in a parallel manner. In a recent empirical study, Aschauer (1988) argues that the services from government infrastructure are particularly important in this context” (bold is ours).

The interplay between private and public capital arises in two different ways in our model. The first follows immediately from the properties of a Cobb-Douglas production function. In Eq. (1’) it is evident that the two inputs are complementary for each other in the production of the final good \( Y \) since, for given \( K \), an increase in \( G_t \) raises the marginal productivity of private capital. Likewise, for given \( G_t \), an increase in \( K \) increases the marginal productivity of public capital. This is the traditional complementarity–effect across inputs that derives from employing a Cobb-Douglas

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4 The closed-economy assumption is sensible given our focus on (infrastructure) public capital, typically not internationally mobile.
5 In the remainder of the paper, public capital, infrastructure capital and infrastructure public capital are synonyms and, thus, will be used interchangeably.
aggregate production function. Arrow and Kurz (1970) were among the first to introduce public capital as an input, complementary for private capital, into an aggregate production function.

Following the pioneering contributions by Aschauer (1989) and Munnell (1990, 1991, 1992), a rapidly-increasing literature has highlighted the importance of public capital stock in raising the productivity of private capital. Recent studies by Ramirez (2000), Ramirez and Nazmi (2003) and Erden and Holcombe (2005), taking into account the distinction between developed and developing countries, find clear evidence for crowding-in effects of public on private capital and stress the importance of distinguishing between the two main components (productive and unproductive) of public capital. In particular, public investment in infrastructure is found (Cohen and Paul, 2004; Erden and Holcombe, 2005) to have a positive and significant impact on private activity through the rise of the marginal productivity of private capital and/or the reduction of private costs.

However, and this is the most important novelty of our contribution, we also allow for another possible interaction between public and private capital. We assume that (depending on the sign of a key-parameter) public and private capital investments can be either complementary or substitutes for each other. More formally, we postulate that the law of motion of public capital is given by:

\[ G_t = (1 - s_g) G_{t-1} + \varphi \gamma p K_t, \quad -\infty < \varphi < 1, \quad \varphi \neq 0. \]  

Eq. (2) represents the main departure of our model from the standard public capital–augmented endogenous growth framework. In this equation \( G \) denotes the evolution over time of the aggregate public capital stock, \((1 – s_g)\) is the share of this stock devoted to accumulating furthermore the existing stock of public capital and \( \gamma p K \equiv K/K \) is the growth rate of private physical capital. The term \((1 – s_g)G\) in Eq. 2 captures the presence of a sort of intertemporal externality arising from the existing stock of public capital: ceteris paribus, the larger \( G \) the lower the fraction of it, i.e. \((1 – s_g)\), that needs to be used in order to increase the actual stock of public capital by a given amount in the time unit. The term \( \varphi K (G/K) \) has been added to illustrate the role of private capital investment in the provision of new productive public capital. This term suggests that, for given \( G \) and \( K \), a change in the rate of investment in private capital affects proportionally the rate of investment in public capital. The parameter \( \varphi \) represents the factor of proportionality between the two investment rates. Since we allow \( \varphi \) to be lower than one and different from zero, we postulate a “technology” for public capital accumulation where, all the rest remaining equal, a faster investment in private capital may either foster \((0 < \varphi < 1)\) or deter \((-\infty < \varphi < 0)\) the rate at which public capital accumulates over time.\(^6\)

Thus, and as a whole, in the present paper we consider the existence of two different possible sources of complementarities between the two reproducible inputs: the first involves their stocks and takes place in output-production activity, whereas the second concerns, eventually, the rates of investment in the two forms of capital.\(^7\) The complementarity between the two kinds of investment may be viewed as the consequence, for example, of the increased demand (and, thus, supply) for public infrastructures following the decision by private agents to increase their own stock of physical capital. Likewise, in the presence of a negative \( \varphi \), Eq. (2) is able to capture the existence of substitutability between public and private capital investments: ceteris paribus, the more negative \( \varphi \),

\(^6\) Proposition 1 will explain more carefully why we require \(-\infty < \varphi < 1\) and \( \varphi \neq 0 \). Clearly, with \( \varphi = 0 \) we lose any interaction between the investments in the two types of capital stock.

\(^7\) While the relation of complementarity between the stock of private capital and the stock of public capital arises in the model from the peculiar type of aggregate production function used (Cobb-Douglas vs. CES), the potential relation of complementarity between the investments in the two types of capital depends, instead, on the sign of \( \varphi \).
the more negative the effect of an increase in $\dot{K}$ on $\dot{G}$ (in this case private capital investment crowds-out public capital investment).

From Eqs. (1') and (2) together we argue that at each time $t \geq 0$ a fraction ($s_y$) of the available aggregate stock of public capital ($G$) is devoted to production of consumption goods and the remaining fraction ($1 - s_y$) is employed to accumulate new public capital. Thus, we have:

$$G_y \equiv s_y G$$

$$G_G \equiv (1 - s_y) G.$$

This formulation introduces some important differences with respect to Barro (1990). Unlike Barro (1990), we explicitly assume that $G$ takes the form of a stock (as opposed to a flow-variable), that the provision of new public capital ($G$) uses public capital as an input, and that such a provision is subject to the influence (whose sign is dictated by the sign of $\phi$) of private incentives to invest in physical capital (as summarized by $\gamma_K$): the larger (more positive) $\phi$, the larger (more positive) the effect of an increase of private capital investment on public capital accumulation.

Empirical evidence supporting Eq. (2) is mainly based on the time-series properties of public and private capital accumulations. Erenburg and Wohar (1995), using US data for the period 1954-1989, examine (within a Granger-causality framework) the relation between private investment and public capital. By considering the public capital stock as a productive input, they show that increases in the private capital stock have a beneficial effect on the stock of public capital in the context of cash flow and Tobin’s $q$ models. More recently, the issue of complementarity between the two forms of investment has been addressed using a panel of 25 developing countries from 1980 to 2003 (Atukeren, 2005). The main result of this study stems from the long-term co-integration analyses and formal Granger-causality tests: of the 25 countries examined, private investment crowds-in public investment in eleven countries and displays a crowding-out effect in solely two. The fact that, in general, Granger-causality tests allow not only for crowding-in (complementarity), but also for crowding-out (substitutability) effects justifies the fact that in the model $\phi$ can be either positive or negative.

At this stage two further comments need to be made about Eq. (2). The first relates our model to the branch of new growth theory that makes depreciation rates endogenous. At this aim, consider a situation in which $\phi$ is negative (private and public capital investments are substitutes for each other): in this case the larger (more negative) $\phi$, the more adverse the effect of an increase of private capital investment on the accumulation of public capital. Since $\gamma_K$ is endogenous, in this situation public capital would be characterized along the BGP by a constant and endogenous depreciation rate, given by the interaction of an exogenous parameter ($\phi$) with an endogenous variable (the growth rate of private capital). The second comment concerns our choice of including in Eq. (2) the growth rate of private capital ($\gamma_K \equiv \dot{K}/K$), rather than $\dot{K}$. The main reason behind this choice has to do with the fact that in this paper we perform BGP analysis and focus on a long-run equilibrium (the BGP equilibrium) in which all variables depending on time grow at constant, possibly positive exponential rates (in a moment we shall introduce a more formal definition of this equilibrium). To be more precise, suppose that Eq. (2) were recast as:

$$\dot{G}_t = (1 - s_y) G_t + \phi K_t G_t,$$

which also implies:

$$s_y = \phi \gamma_K K_t + (1 - \gamma_G),$$

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8 We are considering productive government spending in assets (public infrastructures) that may be used in part as an input to the production of consumption goods and in part to increase the existing stock of public infrastructural capital. We assume that at each point in time the existing stock of public capital can always be clearly split between its two alternative uses.
where \( \gamma_k \equiv \frac{\dot{K}_t}{K_t} \), \( \gamma_G \equiv \frac{\dot{G}}{G} \) and \( K_t = K(0)e^{\gamma_t} \). After taking the derivative of \( s_y \) with respect to time:

\[
\frac{\partial s_y}{\partial t} = \varphi \gamma^2_k K(0)e^{\gamma_t} = \varphi \gamma^2_k K_t ,
\]

and computing

\[
\lim_{t \to +\infty} s_y = \varphi \gamma_k K(0) \lim_{t \to +\infty} e^{\gamma_t} ,
\]

we notice that \( s_y \) is either always increasing (when \( \varphi > 0 \)), or always decreasing (when \( \varphi < 0 \)) over time \( (t \geq 0) \) for each \( \gamma_k > 0 \) and \( K(0) > 0 \). Moreover, at \( t \to +\infty \), \( s_y \) goes either to \( +\infty \) (\( \varphi > 0 \)) or to \( -\infty \) (\( \varphi < 0 \)) for each \( \gamma_k > 0 \) and \( K(0) > 0 \). Since it is a share, in the first case \( \varphi > 0 \) \( s_y \) would reach one, while in the second case \( \varphi < 0 \) \( s_y \) would reach zero in finite time. In other words, \( G_t = (1-s_y)G_t + \varphi \dot{K}_t G_t \) is not compatible with a long-run equilibrium \( (i.e., t \to +\infty) \) in which the factor-input \( G \) is simultaneously employed in part to produce consumption goods and in part to increase the existing stock of public capital.

In brief, the model’s main features are captured by the aggregate production function and the laws of motion of the two reproducible inputs, namely public \((G)\) and private \((K)\) capital:

\[
Y_t = K_t^{1-\alpha}G^\alpha_y , \quad \alpha \in (0,1) , \quad G_y \equiv s_y G_t, \quad (1')
\]

\[
\dot{K}_t = Y_t - C_t - \dot{G}_t , \quad (3)
\]

\[
G_t = (1-s_y)G_t + \varphi \gamma_k G_t , \quad -\infty < \varphi < 1 , \quad \varphi \neq 0 . \quad (2)
\]

In Eq. \((1')\) private capital displays positive but diminishing marginal productivity. The possibility of generating positive growth in the long-term arises because government activity acts as a countervailing-force on the diminishing marginal returns of private capital in goods production. Eq. \((3)\) reflects the aggregate resource-constraint for a closed economy: total output \((Y)\) can be in part consumed \((C)\), in part invested in the form of private capital \((I_k = \dot{K} + \delta_k K)\), and in part invested in the form of public capital \((I_G = G + \delta_G G)\), where \( I_x \) is gross investment in capital of type \( \equiv K, G \). For the sake of simplicity, we assume \( \delta_k = \delta_G = 0 \), with \( \delta_x \) denoting the instantaneous obsolescence rate of capital \( X \).

A benevolent social planner who seeks to maximize intertemporally the discounted instantaneous utility attained by an infinitely lived representative worker/consumer has to decide how to allocate \( G \) across sectors. In the benchmark model (the first we present), this decision is taken as predetermined or exogenous, while in a more realistic setting \( s_y \equiv G_t / G \) is optimally chosen. In the latter case, at equilibrium it must be true that the marginal productivity (in value) of \( G \) used as an input into final output production, that is \( G_y \), must equal the marginal productivity (in value) of \( G \) employed in the process of accumulating new public capital, that is \( G_G \equiv (1-s_y)G \). Formally,

\[9\]

\[9\] Insofar as public capital is intended to represent the stock of public infrastructures like highways, airports, electrical and gas facilities, we can easily infer that since this kind of capital has a very long life-span it also has a very negligible instantaneous depreciation rate, \( \delta_x \). Setting \( \delta_x = 0 \), as well, simplifies significantly the analysis.
\[
\frac{\partial Y}{\partial G_Y} P_y = \frac{\partial G}{\partial G_Y} P_G,
\]

where \( P_G \) can be understood as the shadow price of public capital (\( G \)). We know by assumption that \( P_y = 1 \), while (from the law of motion of \( G \)) we obtain: \( \frac{\partial G}{\partial G_Y} = 1 \). Thus, in the end we can derive the following equilibrium expression for the shadow price of public capital in terms of (final) goods:

\[
P_G = \frac{\partial Y}{\partial G_Y}.
\] (4)

Eq. (4) suggests that in the long-run the shadow price of public capital should equal the productivity of this form of capital in terms of goods.

After presenting and discussing the main characteristics of this economy, we can now analyze in detail the first two theoretical models. While in both of them we postulate a Cobb-Douglas production function in the consumption goods sector, in the first model \( s_y \) is taken as given (exogenous and constant), whereas in the second one it will be considered as fully endogenous. The objective of our analysis is a better understanding of the main determinants of the economy’s optimal balanced growth rate, with particular attention to the parameter \( \varphi \) (measuring the degree of complementarity/substitutability between the investments in private and public capital).

3. Model I: Cobb-Douglas Production Function and Exogenous \( s_y \)

In the benchmark case the aggregate production function is Cobb-Douglas and the distribution of public capital between production of goods and provision of new productive public capital (\( s_y \)) is exogenous and constant.

A benevolent social planner maximizes intertemporally the discounted instantaneous utility of a representative infinitely lived worker/consumer attained from the consumption of the homogeneous final good, subject to the aggregate resource-constraint (Eq. 3), the law of motion of public capital (Eq. 2), and the aggregate production function (Eq. 1’):

Max
\[
\int_0^\infty u(C_t) e^{-\rho t} dt,
\]

\[ u(C_t) = \frac{C_t^{1-\theta} - 1}{1-\theta}, \quad \theta > 0, \quad \rho > 0 \]

s.t.: \[ K_t = (s_y G_t)^{\frac{\alpha}{\alpha}} K_{t-1}^{1-\alpha} - C_t - G_t, \]
\[ s_y \in (0,1), \quad \alpha \in (0;1) \] (SP1)

\[ \dot{G}_t = (1-s_y)G_t + \varphi Y_t; G_t, \quad -\infty < \varphi < 1, \quad \varphi \neq 0 \]

\[ \lim_{t \rightarrow \infty} K_t = 0, \quad \lim_{t \rightarrow \infty} G_t = 0 \]

\[ K(0) > 0, \quad G(0) > 0. \]

Since we are normalizing the labor-force/population size (\( L \)) to one, \( C \) represents aggregate/per worker/per capita consumption. We assume that the instantaneous utility function – that relates the flow of utility per person to the quantity of consumption per person, \( u(C) \) – is CRRA with \( \theta^{-1} > 0 \) representing the constant elasticity of intertemporal substitution in consumption. As it is well-known this functional form is general enough and reproduces (as a
special case, when \( \theta = 1 \) a logarithmic felicity function.\(^{10}\) In the social planner’s problem (SP1) we denoted by \( \rho \) the constant pure rate of time preference and by \( \lambda_K \) and \( \lambda_G \) the two co-state variables associated, respectively, to the state variables \( K \) and \( G \). In what follows we solve for the BGP equilibrium of the model, defined as follows:

**DEFINITION: BGP EQUILIBRIUM**
A BGP Equilibrium in this economy is an equilibrium-path along which:

\[
\begin{align*}
&i) \text{ All time-dependent variables grow at constant (possibly positive) exponential rates;} \\
&ii) \text{ The ratio of the two endogenous state variables, } K/G, \text{ remains invariant over time.}
\end{align*}
\]

The reason why we look at a BGP equilibrium in which \( K \) and \( G \) grow at the same constant rate can be explained as follows. Suppose that \( K \) grows faster than \( G \). If this were the case, in the very long-run \( (t \to \infty) \) the size of public sector \((G)\) would tend to become infinitely small with respect to the size of private sector \((K)\). Clearly, the contrary would be true if \( G \) were to grow faster than \( K \) (the size of private sector would become infinitely small with respect to the size of public sector). Since we want to rule out from our analysis these two extreme cases, we focus on a situation in which \( G \) and \( K \) grow at the same constant rate. In the very long-run this assumption would lead to a balanced ratio of private to public sector sizes and to the co-existence of the two forms of capital in goods production.\(^{11}\) We can now state the main results of the benchmark model.

**PROPOSITION 1**
In an economy described by Eqs. (2) and (3), and in which:
- The aggregate production function is Cobb-Douglas (Eq. 1');
- The instantaneous utility function is CRRA;
- The allocation of public capital to goods-production \((0 < s_y < 1)\) is exogenous and constant,
along the BGP the optimal growth rate of the economy \((\gamma)\) is:

\[
\frac{\dot{G}_t}{G_t} = \frac{\dot{K}_t}{K_t} = \frac{\dot{C}_t}{C_t} = \frac{\dot{Y}_t}{Y_t} \equiv \gamma = \frac{1 - s_y}{1 - \varphi}.
\]

**Proof:** Appendix A. □

Eq. (5) reveals that \( \gamma \) is a function of two key exogenous technological parameters, \( s_y \) and \( \varphi \). Because changes in \( s_y \) (that is, in the way public capital is allocated to the production of goods) are able to affect \( \gamma \), we can interpret this model as a *semi-endogenous* one (see Jones, 1995). In general \( \varphi \) can be either positive or negative. However, with \( s_y \in (0,1) \), for \( \gamma \) to be strictly positive we require:

\[
\varphi < 1 \quad \text{and} \quad \varphi \neq -\infty.
\]

\(^{10}\) More formally, the instantaneous utility function can be recast as: \( u(C_t) = \begin{cases} \frac{c_t^{-\varphi} - 1}{-\varphi}, & \varphi > 0, \ \varphi \neq 1 \\ \ln(c_t), & \varphi = 1 \end{cases} \)

\(^{11}\) To keep the average and marginal productivities of private and public capital constant in the long-run, \( G \) and \( K \) have to grow at the same rate.
These restrictions prevent the optimal growth rate of the economy either from exploding \((\varphi = 1)\), or being negative \((\varphi > 1)\), or else being equal to zero \((\varphi = -\infty)\). Although \(\varphi = 0\) is plausible in Eq. (5), we ignore this case because we are interested in analyzing the growth-effects of any possible interaction between the investments in private and public capital.

**PROPOSITION 2**

*In an economy described by Eqs. (2) and (3), and in which:*
- The aggregate production function is Cobb-Douglas (Eq. 1’);
- The instantaneous utility function is CRRA;
- The allocation of public capital to goods-production \((0 < s_y < 1)\) is exogenous and constant,

the optimal balanced growth rate of the economy \((\gamma)\) depends negatively on \(s_y\), and positively on \(\varphi\).

**Proof:** Immediate from Eq. (5). ■

The first part of Proposition 2 concerns the role of \(s_y\) in economic growth: along the BGP an increase in the share of public capital going to the production of final output by lowering \((1 - s_y)\) decreases, ceteris paribus, the investment in \(G\) and thus the optimal growth rate.\(^{12}\) In other words, by using more public capital into current production, the economy is reducing the amount of an input into the accumulation of public capital for the future, so hampering long-run growth. The second part of the Proposition focuses on the growth effects of \(\varphi\): the larger and more positive is the magnitude of this parameter (i.e., the more complementary private and public capital investments), the larger and more positive is the effect that any increase in private capital accumulation has on public capital investment and, hence, on the optimal growth rate of the economy.

To sum up, in the case of a constrained economy (an economy whose social planner takes \(s_y\) as a constant and exogenously given parameter), we see that the optimal growth rate \(\gamma\) depends linearly on both \(s_y\) (the way public capital is allocated between production of goods and investment in new public infrastructure) and \(\varphi\) (a measure of the degree of complementarity/ substitutability between public and private capital investments). In the next section we analyze the predictions of a more realistic model in which the benevolent social planner takes \(s_y\) as a fully endogenous variable.

### 4. Model II: Cobb-Douglas Production Function and Endogenous \(s_y\)

A benevolent social planner now solves:

\[
\begin{align*}
\max_{\{C_t,\tilde{y}_t, \tilde{K}_t, \tilde{G}_t\}} \int_0^\infty u(C_t) e^{-\rho t} dt, \quad & u(C_t) = \frac{C_t^{\theta} - 1}{1 - \theta}, \quad \theta > 0, \quad \rho > 0 \\
\end{align*}
\]

\(^{12}\)Total output can be recast as a linear function of \(G\), that is: \(Y = \tilde{A}G\), where \(\tilde{A} = (G/K)^{\alpha(1-\alpha)} s_y^\alpha\). Along the BGP \(s_y\) is exogenous and constant and \(G/K\) is also constant, though endogenous. Hence, this is an “\(\tilde{A}G\)” model where the growth rate of output, \(Y\), equals the growth rate of public capital stock, \(G\).
\[ s.t.: \quad \dot{K}_t = s_t G_t^\alpha K_t^{1-\alpha} - C_t - \gamma_t, \quad \alpha \in (0;1) \quad (SP2) \]
\[ \dot{G}_t = (1-s_t) G_t + \phi \gamma_t K_t, \quad -\infty < \phi < 1, \quad \phi \neq 0 \]
\[ \lim_{\lambda \to \infty} \lambda K_t = 0, \quad \lim_{\lambda \to \infty} \lambda G_t = 0 \]
\[ K(0) > 0, \quad G(0) > 0. \]

Unlike the benchmark case, \( s_t \) is a choice variable for the decision-maker. Hence, in the problem (SP2) there are two state variables (\( K \) and \( G \)) and two control variables (\( C \) and \( s_t \)). Since the aggregate production function exhibits constant returns to scale to private and public capital and the sector producing final goods is populated by many atomistic small firms, in equilibrium the two factor-inputs (\( G \) and \( K \)) will receive their own marginal product (in terms of final output, \( Y \)):
\[ P_G = \frac{\partial Y}{\partial G} = \alpha \left( \frac{K_t}{s_t G_t} \right)^{1-\alpha}, \quad (4') \]
\[ r = \frac{\partial Y}{\partial K} = (1-\alpha) \left( \frac{s_t G_t}{K_t} \right)^{\alpha}. \quad (6) \]

In Eqs. (6) and (4'), \( r \) and \( P_G \) stand, respectively, for the rental and shadow prices of private and public capital (the rate of return on the two forms of capital in terms of the numeraire good). We continue to use the same definition of BGP equilibrium given earlier. Appendix B provides formal resolution of this optimization problem. The next theorems and lemma summarize, instead, the main results.

**Theorem 1: Optimal Balanced Growth Rate**

In an economy described by Eqs. (2) and (3), and in which:
- The aggregate production function is Cobb-Douglas (Eq. 1');
- The instantaneous utility function is CRRA;
- The allocation of public capital to goods-production (\( s_t \)) is endogenous,
the positive optimal balanced growth rate (\( \gamma > 0 \)) of the economy is implicitly given by:
\[ \alpha (\rho + \theta \gamma) + \left( \frac{1}{1-\alpha} \right)^{1-\alpha} (\rho + \theta \gamma)^{\alpha} = \alpha(1 + \phi \gamma). \quad (7) \]

*Proof: Appendix B.*

In principle, solving Eq. (7) in terms of \( \gamma \) would yield the solution to the benevolent social planner’s problem (SP2). Although this equation cannot in general be solved in closed form,\(^{13}\) it is evident that the economy’s optimal balanced growth rate depends implicitly on the preference (\( \rho \) and \( \theta \)) and the technological (\( \alpha \) and \( \phi \)) parameters of the model.

\(^{13}\) Eq. (7) admits an ‘easy’ closed-form solution, i.e. \( \gamma = (1-2\rho)/(2\theta - \phi) \), if we let \( \alpha \) tend to one. In this case the production function would be: \( Y = s_t G \). This amounts to saying that the economy produces with an “AG”-type aggregate technology in which private physical capital, \( K \), is replaced by public capital, \( G \). Even though the assumption \( \alpha \to 1 \) allows obtaining a closed-form result, we prefer to analyze the more general model with both forms of capital being simultaneously used in the production of goods.
**Theorem 2:** Existence and Uniqueness of the Optimal Balanced Growth Rate

In an economy described by Eqs. (2) and (3), and in which:
- The aggregate production function is Cobb-Douglas (Eq. 1');
- The instantaneous utility function is CRRA;
- The allocation of public capital to goods-production \((s_y)\) is endogenous,
a positive optimal balanced growth rate \((\gamma > 0)\) does exist and is unique, provided that:

\[
\alpha(1-\rho) > \left( \frac{1}{1-\alpha} \right)^{\frac{1-\alpha}{\alpha}} \rho^{\frac{1}{\alpha}}.
\]

This restriction is always satisfied, for any admissible value of \(\alpha \in (0;1)\), when \(\rho\) is lower than a threshold level \((\rho < \bar{\rho} \equiv 0.381)\).

Proof: Define \(F(\gamma) \equiv \alpha(\rho + \theta \gamma) + \left( \frac{1}{1-\alpha} \right)^{\frac{1-\alpha}{\alpha}} (\rho + \theta \gamma)^{\frac{1}{\alpha}}\) and \(G(\gamma) \equiv \alpha(1+\varphi \gamma)\). Thus, Eq. (7) can be recast as: \(F(\gamma) = G(\gamma)\). A positive optimal balanced growth rate \((\gamma > 0)\) does exist and is unique when a positive intersection between functions \(F(\gamma)\) and \(G(\gamma)\) does exist and is unique. Both functions are continuous in \(\gamma\). Function \(F(\gamma)\) is increasing and convex in \(\gamma\), whereas function \(G(\gamma)\) depends linearly on \(\gamma\) (its slope, \(a \varphi\), is constant and can be either positive or negative depending on the sign of \(\varphi\)). Moreover, we also observe that: \(\lim\limits_{\gamma \rightarrow +\infty} \frac{\partial F(\gamma)}{\partial \gamma} = +\infty\); \(F(0) = \alpha \rho + \left( \frac{1}{1-\alpha} \right)^{\frac{1-\alpha}{\alpha}} \rho^{\frac{1}{\alpha}} \equiv \psi > 0\), and \(G(0) = \alpha > 0\). Given all this, a simple (sufficient) condition for a positive intersection between functions \(F(\gamma)\) and \(G(\gamma)\) to exist and to be unique (for any \(-\infty < \varphi < 1\), \(\varphi \neq 0\) ) is, therefore, \(\alpha = G(0) > F(0) = \psi\), or \(\alpha(1-\rho) > \left( \frac{1}{1-\alpha} \right)^{\frac{1-\alpha}{\alpha}} \rho^{\frac{1}{\alpha}}\). It is possible to show numerically that this inequality is satisfied, for each admissible value of \(\alpha \in (0;1)\), when \(\rho\) is lower than a threshold level \((\rho < \bar{\rho} \equiv 0.381)\).

Empirical evidence suggests that the pure rate of time preference is clearly below \(\bar{\rho} \equiv 0.381\). For example, Pearce and Ulph (1999) suggest that a figure close to 1.5% would be suitable for \(\rho\) and Barro and Sala-i-Martin (2004, p. 260) consider precisely the case \(\rho = 2\%\) as “standard”.\(^{14}\)

From this, we may safely infer that a positive optimal balanced-growth rate \((\gamma > 0)\) does exist and is unique in our setting.

Thus, with Cobb-Douglas aggregate production function in the final output sector moving from a model in which the sectoral distribution of public capital (between production of goods and provision of new infrastructure capital) is exogenous to an identical model where it is fully endogenous, the long-run optimal balanced growth rate of the economy turns out to depend not only

\(^{14}\)Creedy and Guest (2008, Fig.1, p. 111) also consider extremely low values of \(\rho\) (in the range 0.2% - 3.8%) in their simulation exercises.
on \( \varphi \), but also on the preference parameters (\( \rho \) and \( \theta \)). In addition, it becomes a function of the public capital share in total GDP (\( \alpha \)).\(^{15}\) The following theorem analyzes in more detail the impact of \( \varphi \) on \( \gamma \) in the presence of Cobb-Douglas aggregate production function and endogenous \( s_y \).

**THEOREM 3:** The effect of an increase in \( \varphi \) on the optimal balanced growth rate

In an economy described by Eqs. (2) and (3), and in which:

- The aggregate production function is Cobb-Douglas (Eq. 1');
- The instantaneous utility function is CRRA;
- The allocation of public capital to goods-production (\( s_y \)) is endogenous,

an increase in \( \varphi \) exerts a positive effect on the optimal balanced growth rate of the economy (\( \gamma > 0 \)) as long as:

\[
\theta > \max\{0; \varphi\}.
\]

Proof: After some algebra, implicit differentiation of Eq. (7) with respect to \( \varphi \) yields:

\[
\frac{\partial \gamma}{\partial \varphi} = \frac{\alpha \gamma}{\alpha (\theta - \varphi) + \left(\frac{1}{1- \alpha}\right)^{\frac{1-a}{\alpha}} \frac{\theta}{\alpha} (\rho + \theta \gamma)^{(\frac{1-a}{\alpha})}}.
\]

With \( \gamma > 0 \) and \( \theta > 0 \), a simple condition for the right hand side of this expression to be always positive is \( \theta > \varphi \). Note that \( \theta \geq 1 \) and \(-\infty < \varphi < 1\) easily meet the condition \( \theta > \max\{0; \varphi\} \). Appendix B shows that \( \theta \geq 1 \) also guarantees that the two transversality conditions do hold along the BGP equilibrium.

**LEMMA 1:** The effect of an increase in \( \varphi \) on the optimal balanced growth rate with logarithmic instantaneous preferences

In an economy described by Eqs. (2) and (3), and in which:

- The aggregate production function is Cobb-Douglas (Eq. 1');
- The instantaneous utility function is logarithmic;
- The allocation of public capital to goods-production (\( s_y \)) is endogenous,

an increase in \( \varphi \) always exerts a positive effect on the optimal balanced growth rate (\( \gamma > 0 \)) of the economy.

Proof: With logarithmic instantaneous utility function, \( \theta = 1 > \varphi \in (-\infty; 1) \).

Hence, unlike the case of exogenous \( s_y \in (0; 1) \), in which an increase of \( \varphi \) unambiguously raises \( \gamma \), when \( s_y \) is optimally chosen for \( \varphi \in (-\infty; 1) \) to exert a positive effect on the optimal economic growth rate it must be (sufficient condition) \( \theta > \max\{0; \varphi\} \). This restriction is trivially checked under instantaneous logarithmic preferences (Lemma 1). However, and apart from the theoretically very special case of logarithmic preferences, empirical evidence too points out that \( \theta > \max\{0; \varphi\} \) is actually the most relevant case. In this respect, it is worth recalling here that empirical work on \( \theta \) involves three main alternative and fundamentally different approaches. On

\(^{15}\) We shall come back to this issue in Section 6.
the one hand, micro models of *lifetime consumption behavior* (Blundell et al., 1994 and Attanasio and Browning, 1995, among others) obtain country estimates of $\theta$ that are close (and normally just above) to unity. On the other hand, models based on the *market demand for preference-independent goods* (food is generally considered to be the best example of such goods) include several recent studies for different countries and use co-integration techniques (examples in this tradition are Evans, 2004 and Percoco, 2008). These studies all suggest that $\theta$ is definitely closer to 1.5 than to 1. An alternative approach to the estimate of the elasticity of marginal utility of consumption is based on the *revealed social values of governments* (for instance Cowell and Gardiner, 1999; Evans and Sezer, 2004). According to this method a suitable value for $\theta$ may be inferred directly from government spending/tax policies: the extent of progressiveness in a country’s personal income tax rates, the advocates of this approach argue, reveals in fact a government’s degree of aversion to income inequality. In this line of research Evans and Sezer (2004) find that for six major OECD countries, estimates of $\theta$ are in the range 1.3 to 1.6, with a value of 1.5 for the UK.\footnote{New evidence presented for 20 OECD countries using income tax rates on gross wage earnings of single persons without dependants in 2002 (Evans, 2005) suggests that, on average, for the full sample of countries $\theta$ is close to 1.4. In line with models based on *lifetime consumption behavior*, Beaudry and van Wincoop (1996) and Guvenen (2006) also suggest that $\theta$ is not significantly different from one in the data. In general, empirical investigations bring convincing evidence that the case $\theta<1$ appears as very unlikely to occur (Hall, 1988; Patterson and Pesaran, 1992 and Favero, 2005, among others, all point to a value of $\theta$ definitely larger than one). Using Japanese aggregate data, Okubo (2009) has recently showed that the point-estimates of the elasticity of substitution in consumption for Japan are around 0.2 - 0.4, a value not significantly different from the one that can be found for the US.}

With $\theta > \text{Max}\{0; \varphi\}$ being not only theoretically but also empirically defendable, we conclude that in the model analyzed so far an increase of $\varphi$ yields a positive impact on the optimal balanced growth rate ($\gamma>0$). In what follows, we extend our analysis by postulating a more flexible and general form of aggregate technology for the production of goods.

### 5. An Extension: CES Aggregate Technology in the Production of Final Output

In the previous sections we have been using the assumption that the aggregate technology for goods production is Cobb-Douglas, that is:

$$Y_t = K_t^{-\alpha} G_t^{\alpha}, \quad \alpha \in (0;1).$$

This formulation is the one used by Barro (1990), with $A=1$, in his celebrated growth paper with productive government expenditure.\footnote{Barro (1990, Eq. 10, p. S107) uses the following Cobb-Douglas production function: $y = A g^{k^{1-a}}$. In his words (pp. S106-S107): “...The variable $k$ is the representative producer's quantity of capital, which would correspond to the per capita amount of aggregate capital. I assume that $g$ can be measured correspondingly by the per capita quantity of government purchases of goods and services”. In Barro (1990) the whole amount of $g$ is employed in sector $y$.} In this paragraph we extend our previous analysis to the case of a more general *CES* (Constant Elasticity of Substitution) aggregate production function. Many reasons justify our interest in this extension. The first is mainly theoretical in nature. We now know that the elasticity of factor-substitution plays a relevant role in economic growth.\footnote{According to de La Grandville (2009), it is one of the main determinants of the level of economic growth. It also affects the behavior of the savings rate along the transition path (Smetters, 2003).} In this respect, since the Cobb-Douglas specification of the aggregate production function describes only a very peculiar situation (that in which the elasticity of factor-substitution equals one), modeling aggregate output as generated by a Cobb-Douglas technology may appear as unduly restrictive in many instances. Because the CES production function allows the elasticity of substitution between factors to take a continuum of values (being either greater, or lower, or else equal to one), many papers in the literature\footnote{See, as another example, Miyagiwa and Papageorgiou (2003).} have started to investigate in detail the role of this production function in growth theory. The second reason why it might be interesting to analyze the case of a CES aggregate production function is more empirical. It is well known that, under perfect competition,
the Cobb-Douglas production function is the only production function with the property that relative factor shares are independent of relative factor prices. This implies that “...we should expect zero variation in relative factor shares, when comparing countries at different stages of development. This implication can be resoundly rejected: factor shares do vary from country to country. For example,...labor’s share falls in a range from 0.53 (Venezuela) to 0.78 (Sri Lanka). Therefore, for the purpose of applied work, a more general CES specification would be a better choice, since it can be consistent with this dimension of the cross-country data” (Aiyar and Dalgaard, 2009, p. 290). Another empirical argument in favor of a more general CES production function has to do with the point-estimate of the elasticity of factor substitution. Empirical studies using single-country or microeconomic data to estimate, for example, the elasticity of substitution between capital and labor (Antràs, 2004; Chirinko, 2008) find that such elasticity is well below unity. On the contrary, using aggregate data on a panel of 82 countries over 28 years (from 1960 to 1987) and a variety of different regression-model specifications, Duffy and Papageorgiou (2000) find empirical support in favor of an elasticity of substitution between capital and labor (or effective labor) being significantly greater than unity. Irrespective of the data and the econometric techniques used the point remains that, when modeling the production relationship between inputs (mainly capital and labor) and output, the use of a more flexible CES specification might be preferable.

Therefore, even in the absence (as far as we know) of specific estimates of the elasticity of substitution between private and public capital in the production of goods, in what follows we accept the invitation of Duffy and Papageorgiou (2000, p. 112) and analyze how our results would change under the following specification of the aggregate technology:

\[
Y_i = \left[ (1-\alpha)K_i^\varepsilon + \alpha G_i^\varepsilon \right]^{1/\varepsilon}, \quad \alpha \in (0,1), \quad \varepsilon \leq 1, \quad \varepsilon \neq 0. \tag{8}
\]

This equation is the standard ACMS-version of the CES production function due to Arrow et al. (1961). Note that, to be consistent with the previous sections, we continue here to normalize the TFP parameter \((A)\) to one. Eq. (8) displays constant returns to scale in \(K\) and \(G_i\) together and diminishing marginal returns in \(K\) and \(G_i\), in isolation. While \(\alpha\) and \((1-\alpha)\) continue to be the shares of, respectively, \(G_i\) and \(K\) in GDP, the parameter \(\varepsilon\) determines the elasticity of substitution \((\sigma)\) between the two inputs, with \(\sigma = 1/(1-\varepsilon)\). The higher “flexibility” of the CES-production function with respect to the Cobb-Douglas derives from the fact that, depending on the value of \(\varepsilon\), Eq. (8) allows a clear distinction between three extreme cases. The first one is when \(\varepsilon = 1\). This is the case of perfect substitutability between the two forms of capital (linear production function, \(\sigma = \infty\)). Instead, when \(\varepsilon = -\infty\), there exists no substitutability at all between \(K\) and \(G_i\) (Walras-Leontief production function, \(\sigma = 0\)). Finally, an intermediate case between the previous two is when \(\varepsilon = 0\). In this case \(\sigma = 1\) and, as already mentioned, the aggregate production function (8) is Cobb-Douglas (Eq. 1’). Since we have already analyzed what happens with Cobb-Douglas technology we assume \(\varepsilon \neq 0\). Moreover, although in its most general formulation \(\varepsilon\) can equal 1 and \(-\infty\), it is convenient (not only for the sake of generality, but also to avoid possible case-distinctions) to ignore these two opposite situations, as well. Thus, in the remainder of the paper we focus on values of \(\varepsilon\) belonging to the interval: \(\varepsilon \in (-\infty;0)\cup(0;1)\).

We are now ready to summarize the main results of the model with CES-technology in the sector producing the homogeneous consumption goods. As before, we start by considering the

20 Indeed, this is probably the main reason why economists (especially macroeconomists) have been using the Cobb-Douglas so extensively in the last decades.

21 The tendency of factor shares to vary has also been put forward by Blanchard (1997) and Bentolila and Saint-Paul (2003).

22 “...It is our hope that the findings reported in this article will encourage other researchers, both theoretical and applied, to consider the more general CES specification for the aggregate input-output production relationship...when constructing models of economic growth...”.

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(simplest) situation of a *constrained economy* (an economy that takes \( s_y \) as a constant and exogenously given parameter).

**Proposition 3**

In an economy described by Eqs. (2) and (3), and in which:
- The aggregate production function is CES (Eq. 8);
- The instantaneous utility function is CRRA;
- The allocation of public capital to goods-production \((0 < s_y < 1)\) is exogenous, along the BGP the optimal growth rate of the economy \((\gamma)\) is given by:

\[
\dot{G}_t = \frac{\dot{K}_t}{G_t} = \frac{\dot{C}_t}{C_t} = \frac{\dot{Y}_t}{Y_t} \equiv \gamma = \frac{1-s_y}{1-\varphi} \tag{9}
\]

**Proof:** Appendix A. ■

Regarding Eq. (9) a comment concerning the role of the parameter determining the elasticity of factor-substitution \((\varepsilon)\) is in order. We clearly see from this equation that \(\gamma\) is unaffected by \(\varepsilon\) and continues to depend solely on \(\varphi\), for given \(s_y\). Because \(\gamma\) is independent of \(\varepsilon\), Eq. (9) coincides with Eq. (5) found in the Cobb-Douglas case. In other words, in a *constrained economy* the optimal growth rate turns out to be independent of the type of aggregate technology employed for producing consumption goods and depends solely on \(\varphi\).

In the next theorems and lemma we analyze the main results of the model under a CES production function with endogenous \(s_y\). Before doing this we see that with CES-technology the productivities of the two factor inputs in goods-production (respectively, \(P_G\) and \(r\)) are:

\[
P_G \equiv \frac{\partial Y_t}{\partial G_{Yt}} = \alpha \left[ (1-\alpha) \left( \frac{K_t}{s_y G_t} \right)^\varepsilon + \alpha \right]^{\frac{1-\varepsilon}{\varepsilon}} \tag{4”}
\]

\[
r = \frac{\partial Y_t}{\partial K_t} = (1-\alpha) \left[ (1-\alpha) + \alpha \left( \frac{s_y G_t}{K_t} \right) \right]^{\frac{1-\varepsilon}{\varepsilon}} \tag{6’}
\]

**Theorem 4:** Optimal Balanced Growth Rate

In an economy described by Eqs. (2) and (3), and in which:
- The aggregate production function is CES (Eq. 8);
- The instantaneous utility function is CRRA;
- The allocation of public capital to goods-production \((s_y)\) is endogenous,

the positive optimal balanced growth rate \((\gamma > 0)\) of the economy is implicitly given by:

\[
\left[ (\rho + \theta \gamma) \varepsilon^{(1-\varepsilon)} - (1-\alpha)^{1/(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}} = \alpha^{\alpha^\varepsilon} \left[ (1-\rho) - (\theta - \varphi) \gamma \right]. \tag{10}
\]

**Proof:** Appendix C. ■
Given our assumptions on the admissible range of $\varepsilon$, the left hand side of Eq. (10) is always definite when:

$$ (\rho + \theta \gamma)^\varepsilon > (1-\alpha). \quad (11a) $$

When (11a) is met, the left hand side of Eq. (10) is always positive, too. Thus, in this case the right hand side of Eq. (10) should be positive, as well. This implies:

$$ 0 < \gamma < \frac{1-\rho}{\theta-\phi}, \quad 0 < \rho < 1 \quad \text{and} \quad \theta > \phi. \quad (11b) $$

It is evident that, in the neighborhood of $\varepsilon = 0$ (Cobb-Douglas case), inequality (11a) is satisfied for every $\rho > 0$, $\theta > 0$, $\gamma > 0$ and $\alpha \in (0;1)$. Likewise, this inequality is met when we let $\alpha$ tend to one.\(^{23}\)

In principle, solving Eq. (10) in terms of $\gamma$ would deliver the constant and positive optimal balanced growth rate of this economy. Unfortunately, in general this equation cannot be solved in closed form in $\gamma$.\(^{24}\) However, we clearly see that $\gamma$ depends implicitly on the preference ($\rho$ and $\theta$) and technological ($\varepsilon$, $\alpha$ and $\phi$) parameters of the model.

**Theorem 5:** Existence and Uniqueness of the Optimal Balanced Growth Rate

In an economy described by Eqs. (2) and (3), and in which:

- The aggregate production function is CES (Eq. 8);
- The instantaneous utility function is CRRA;
- The allocation of public capital to goods production ($s_r$) is endogenous;
- Inequality (11a) is satisfied,

a positive optimal balanced growth rate ($\gamma > 0$) does exist and is unique, provided that the following (sufficient) conditions are checked:

$$ 0 < \rho < 1 \quad (T5.1) $$

$$ \theta > \phi \quad (T5.2) $$

$$ \rho^\varepsilon > (1-\alpha) \quad (T5.3) $$

$$ \alpha^{1/\varepsilon} (1-\rho) > \left[ \rho^{\varepsilon(1/\varepsilon)} - (1-\alpha)^{1/(1-\varepsilon)} \right]^{1-\varepsilon}. \quad (T5.4) $$

\(^{23}\) The term $(\rho + \theta \gamma)$ is known in the literature as Social Time-Preference Rate (STPR), or Consumption Rate of Interest (CRI). According to Pearce and Ulph (1999) a range of 2% - 4% for the STPR is appropriate. Using a tax-based estimate of 1.35 for $\theta$, Evans (2005, Table 6, p. 219) finds that for five major OECD countries (France, Germany, Japan, UK and USA), the STPR ranges between 3.7% (France) and 4.4% (Japan), with an average value of 3.98%. In our model $(1-\alpha)$ is the private capital share in GDP. With $(\rho + \theta \gamma) = 4\%$ and $(1-\alpha) = 1/3$, numerical analysis reveals that $(\rho + \theta \gamma)^\varepsilon > (1-\alpha)$ for each $\varepsilon \leq 0.341303$. With the same figure for $(\rho + \theta \gamma)$ but with $(1-\alpha) = 2/3$, instead, $(\rho + \theta \gamma)^\varepsilon > (1-\alpha)$ holds for each $\varepsilon \leq 0.125964$. Finally, with $(\rho + \theta \gamma) = 4\%$ and $(1-\alpha) = 0.75$ (Barro, 1990, Fig. 1, p. S110), inequality (11a) is met for each $\varepsilon \leq 0.089373$. In general, since $(\rho + \theta \gamma)^\varepsilon$ is decreasing in $\varepsilon$ (for any $\rho + \theta \gamma < 1$), the larger $(1-\alpha)$, the lower the upper bound of $\varepsilon$ that satisfies the constraint $(\rho + \theta \gamma)^\varepsilon > (1-\alpha)$.

\(^{24}\) Indeed, if we let $\alpha$ tend to one Eq. (10) would have the following closed-form solution: $\gamma = (1-2\rho)/(2\theta-\phi)$. Notice that this solution coincides with the one we obtain with endogenous $s_r$ under Cobb-Douglas aggregate production function in the goods sector when $\alpha \to 1$. As before, although the assumption $\alpha \to 1$ allows reaching an ‘easy’ closed-form result, we analyze here the predictions of the most general possible model with both forms of capital being used simultaneously in the final output sector.
Proof: Now define \( F(\gamma) \equiv \left[ (\rho + \theta \gamma)^{\epsilon/(1-\epsilon)} - (1-\alpha)^{1/(1-\epsilon)} \right]^{1-\epsilon} \) and \( G(\gamma) \equiv \alpha^{1/\epsilon} [ (1-\rho) - (\theta - \phi) \gamma ] \). Thus, Eq. (10) implies: \( F(\gamma) = G(\gamma) \). A constant and positive optimal balanced growth rate \((\gamma > 0)\) does exist and is unique when a positive intersection between \( F(\gamma) \) and \( G(\gamma) \) does exist and is unique. When inequality (11a) holds, function \( F(\gamma) \) is always definite for each admissible value of \( \rho \), \( \theta \), \( \gamma \), \( \epsilon \) and \( \alpha \). Moreover, it is increasing in \( \gamma \). Function \( G(\gamma) \), instead, is a straight line being decreasing in \( \gamma \) provided that \( \theta > \phi \). In addition, \( G(0) = \alpha^{1/\epsilon} (1-\rho) \equiv \Omega > 0 \) when \( 0 < \rho < 1 \), and \( F(0) = \left[ (\rho)^{\epsilon/(1-\epsilon)} - (1-\alpha)^{1/(1-\epsilon)} \right]^{1-\epsilon} \equiv \Phi \). For \( \Phi \) to take definite values for any admissible \( \rho \), \( \epsilon \) and \( \alpha \), we require \( \rho^\epsilon > (1-\alpha) \). When this condition holds, then \( \Phi > 0 \). Since \( F(\gamma) \) is increasing and \( G(\gamma) \) is decreasing in \( \gamma \), with \( \Omega \) and \( \Phi \) both positive numbers, a sufficient condition for an intersection between \( F(\gamma) \) and \( G(\gamma) \) in the range \( 0 > \gamma \) to exist and to be unique is: \( G(0) \equiv \Omega > \Phi \equiv F(0) \), implying \( \alpha^{1/\epsilon} (1-\rho) > \left[ (\rho)^{\epsilon/(1-\epsilon)} - (1-\alpha)^{1/(1-\epsilon)} \right]^{1-\epsilon} \). \( \blacksquare \)

Conditions (T5.1) and (T5.2) are always satisfied under plausible values of \( \rho \) and \( \theta \) (i.e., values dictated either by empirical evidence or existing studies – see our previous discussion on the available estimates of these two preference parameters). Moreover, \( \theta > \phi \) always holds under logarithmic instantaneous preferences (Appendix C shows that \( \theta \geq 1 \) also guarantees that the two transversality conditions do hold along the BGP equilibrium). In order to have a rough idea of whether and, eventually, when (T5.3) and (T5.4) are checked we use the same Barro (1990, Fig. 1, p. S110)’s values for \( \alpha \) and \( \rho \) (respectively, \( \alpha = 0.25 \) and \( \rho = 0.02 \)) into these two constraints and notice that they hold simultaneously for each \( \epsilon \leq \bar{\epsilon} \equiv 0.073537929 \), \( \epsilon \neq 0 \). Clearly, the threshold \( \bar{\epsilon} \) is highly sensitive to the values of \( \alpha \) and \( \rho \) used.\(^{25}\) The next theorem analyzes the effect of a rise of \( \phi \) on \( \gamma \) with endogenous \( s_y \) and CES production function in the goods sector.

**Theorem 6:** THE EFFECT OF AN INCREASE IN \( \phi \) ON THE OPTIMAL BALANCED GROWTH RATE

In an economy described by Eqs. (2) and (3), and in which:
- The aggregate production function is CES (Eq. 8);
- The instantaneous utility function is CRRA;
- The allocation of public capital to goods production (\( s_y \)) is endogenous;
- Inequality (11a) is satisfied,

an increase in \( \phi \) exerts a positive effect on the optimal balanced growth rate of the economy \((\gamma > 0)\), provided that:

\[ \theta > \text{Max} \{0; \phi \}. \]

**Proof:** Implicit differentiation of Eq. (10) with respect to \( \phi \), after some algebra yields:

\(^{25}\) As another example, with \( \rho = 0.02 \) and \( \alpha = 2/3 \) (which implies that the share of private capital in GDP equals 1/3) constraints (T5.3) and (T5.4) hold simultaneously for each \( \epsilon \leq \bar{\epsilon} \equiv 0.28082971 \), \( \epsilon \neq 0 \).
\[
\frac{\partial \gamma}{\partial \varphi} = \frac{\alpha^{1-\varepsilon} \gamma}{\Lambda + \alpha^{1-\varepsilon} (\theta - \varphi)}, \quad \Lambda = \theta \left[ (\rho + \theta \gamma)^{\varepsilon/(1-\varepsilon)} - (1 - \alpha)^{\sqrt[1-\varepsilon]{\varepsilon}} \right]^{(1-\varepsilon)^{-1}}. (\rho + \theta \gamma)^{\frac{\varepsilon}{1-\varepsilon}}.
\]

With \( \gamma > 0 \), the numerator of \( \partial \gamma / \partial \varphi \) is always positive. The denominator is always definite and positive when \((\rho + \theta \gamma)^{\frac{\varepsilon}{1-\varepsilon}} > (1 - \alpha) \) and \( \theta > \varphi \), with \( \theta > 0 \).

**Lemma 2:** The effect of an increase in \( \varphi \) on the optimal balanced growth rate with logarithmic preferences

*In an economy described by Eqs. (2) and (3), and in which:*
- The aggregate production function is CES (Eq. 8);
- The instantaneous utility function is logarithmic;
- The allocation of public capital to goods production \((s_y)\) is endogenous;
- Inequality (11a) is satisfied,

an increase in \( \varphi \) always exerts a positive effect on the optimal balanced growth rate of the economy \( \gamma > 0 \).

**Proof:** With logarithmic instantaneous utility function, \( \theta = 1 \) and therefore the condition \( \theta > \operatorname{Max} \{ 0; \varphi \} \), with \( \varphi \in (-\infty; 1) \), is always checked.

Concerning Theorem 6, we have already commented on the restriction \((\rho + \theta \gamma)^{\frac{\varepsilon}{1-\varepsilon}} > (1 - \alpha) \) and on possible combinations of social time-preference rates, \((\rho + \theta \gamma)\), \( \varepsilon \) and \( \alpha \) such that it holds. Likewise, we have already mentioned that \( \theta > \operatorname{Max} \{ 0; \varphi \} \) appears to be easily defensible either on empirical or purely theoretical (logarithmic preferences, Lemma 2) grounds. At the end of this section we want to study the effects of a change in the parameter determining the elasticity of substitution between \( G_y \) and \( K \) in goods production \((\varepsilon)\) on the optimal balanced growth rate of this economy.

**Theorem 7:** The effect of a rise of \( \varepsilon \) on the optimal balanced growth rate

*In an economy described by Eqs. (2) and (3), and in which:*
- The aggregate production function is CES (Eq. 8);
- The instantaneous utility function is CRRA;
- The allocation of public capital to goods production \((s_y)\) is endogenous;
- Inequalities (11a) and (11b) are satisfied,

an increase in \( \varepsilon \in (-\infty; 0) \cup (0; 1) \) has an unambiguously positive effect on \( \gamma \) when the following two (sufficient) conditions hold simultaneously:

\[
(\rho + \theta \gamma)^{\varepsilon/(1-\varepsilon)} > \alpha + (1 - \alpha)^{\sqrt[1-\varepsilon]{\varepsilon}} \tag{T7.1}
\]

\[
\frac{1}{\varepsilon} \left[ (1 - \alpha)^{\sqrt[1-\varepsilon]{\varepsilon}} \ln (1 - \alpha) - (\rho + \theta \gamma)^{\varepsilon/(1-\varepsilon)} \ln (\rho + \theta \gamma) \right] > 0. \tag{T7.2}
\]

An increase in \( \varepsilon \) has an unambiguously negative effect on \( \gamma \) when it is simultaneously true that:

\[
(\rho + \theta \gamma)^{\varepsilon/(1-\varepsilon)} < \alpha + (1 - \alpha)^{\sqrt[1-\varepsilon]{\varepsilon}} \tag{T7.3}
\]

\[
\frac{1}{\varepsilon} \left[ (1 - \alpha)^{\sqrt[1-\varepsilon]{\varepsilon}} \ln (1 - \alpha) - (\rho + \theta \gamma)^{\varepsilon/(1-\varepsilon)} \ln (\rho + \theta \gamma) \right] < 0. \tag{T7.4}
\]
When conditions (T7.1)-(T7.2) or (T7.3)-(T7.4) do not hold simultaneously, the effect of an increase in $\varepsilon$ on $\gamma$ can be either positive, or negative, or else equal to zero.

Proof: Appendix D.

In words, under the assumptions written above, Theorem 7 provides precise conditions for a rise of the elasticity of substitution between public and private capital in goods production (i.e., an increase in $\varepsilon$) to yield either a definitely beneficial, or a definitely detrimental, or else an ambiguous impact on the optimal balanced growth rate of the economy. More specifically, such conditions have to do with the magnitude of three crucial variables of the model, namely the social time-preference rate $(\rho + \theta\gamma)$, the share of public capital in GDP ($\alpha$) and $\varepsilon$ itself.

By comparing the two models with endogenous $s_y$ analyzed so far (the ones with Cobb Douglas and CES technologies, respectively), we conclude (see Eqs. 7 and 10) that optimal growth depends critically also on the degree of substitutability between the two forms of capital used in the final output sector (namely, the parameter $\varepsilon$) and, thus, on the shape of the aggregate production function employed in this industry. The last theorem has showed that the way a change in such elasticity affects the optimal balanced growth rate is a-priori ambiguous and has provided conditions for the relationship between $\gamma$ and $\varepsilon$ to be unambiguous.

The dependence of optimal growth on $\varepsilon$ represents the distinctive feature of the model with CES-production function and endogenous $s_y$ in comparison with the same CES-model but with exogenously given $s_y$. This paper also proved that in the last case, unlike the other one, optimal growth is unambiguously fostered by a higher degree of complementarity between private and public capital investments (i.e., higher $\varphi$).

In the next Section we come back to the case of Cobb-Douglas technology in the sector producing goods. Our objective is to compare Barro (1990)'s prediction concerning the long-run relationship between optimal growth and the share of public capital in GDP with the one coming from our model with endogenous $s_y$.


The model we have presented in Sections 2 through 4 is an extension of the Barro (1990)'s endogenous growth model with public goods. The extension has been twofold. First of all, we have considered productive government expenditure as a stock (public capital), rather than a flow (general governmental purchases of goods and services). Secondly, we postulated that private and public capital are two reproducible productive inputs interacting with each other not only in goods-production but also in terms of their respective investment rates. In this way we were able to study the effects of two distinct sources of interaction between private and public capital: on the one hand, with Cobb-Douglas technology, the stocks of these two capital-inputs are complementary for each other in the production of final output; on the other hand, the investments in the two forms of capital can be either complementary or substitutes for each other. For a better understanding of the major consequences of this modeling strategy on the paper’s results we now compare the salient differences between our and Barro (1990)'s approach concerning the relationship between optimal growth and the share of public expenditure in GDP. We use here the simplified textbook version of Barro (1990) presented in Barro and Sala-i-Martin (2004, pp. 220–223).
In Barro (1990) it is postulated that the aggregate labor force \((L)\) and the level of technology for goods production \((A)\) are constant. If we normalize these two variables to one \((i.e., A_t = L_t = 1 \text{ for each } t)\), the aggregate production function would read as (Barro and Sala-i-Martin, 2004, p. 221, Eq. 4.39):

\[
Y_t = K_t^\alpha G_t^{1-\alpha}, \quad \alpha \in (0;1),
\]

which is similar to our Eq. (1’). The only two differences can be summarized as follows. In our model only the fraction \(s_y\) of the available aggregate stock of public capital \((G)\) is devoted to goods production; moreover, the shares of the two productive factor-inputs in GDP are reversed with respect to those appearing in Eq. (12).\(^{26}\) With \(L = 1,\) per worker \((c \equiv C / L)\) and aggregate \((C)\) consumption levels are clearly the same. Furthermore, if we also use \(A = 1\) and \(\delta = 0\) \((\text{where } \delta \text{ is the instantaneous depreciation rate of } K)\), the economy’s growth rate would equal (Barro and Sala-i-Martin, 2004, Eq. 4.42, p. 221):

\[
\frac{\dot{C}}{C} = \frac{1}{\theta} \left[ \alpha \left( \frac{G}{Y} \right)^{\frac{1-\alpha}{\alpha}} - \rho \right].
\]

A benevolent social planner who seeks to maximize the utility attained by the representative household chooses endogenously the value of the optimal ratio \(G / Y\) and sets this ratio equal to:\(^{27}\)

\[
\frac{G}{Y} = 1 - \alpha.
\]

Hence, the optimal, balanced growth solution is:

\[
\frac{\dot{C}}{C} = \frac{1}{\theta} \left[ \alpha (1-\alpha)^{\frac{1-\alpha}{\alpha}} - \rho \right]. \quad (13)
\]

In order to make a proper comparison with our model, we define:

\[
\beta = 1 - \alpha,
\]

where \(\beta\) is the share of \(G\) in \(Y\). Using this definition into Eq. (13) leads to:

\[
\frac{\dot{C}}{C} \equiv \gamma_c = \frac{1}{\theta} \left( (1 - \beta)^{\frac{\beta}{1-\beta}} - \rho \right). \quad (13')
\]

The following proposition and theorem summarize the main differences between our model with endogenous \(s_y\) and Cobb-Douglas technology in goods-production and Barro (1990)’s as far as the long-run relationship between optimal growth and the share of public capital (spending) in GDP is concerned.

**Proposition 4**

*In Barro (1990) the relationship between the optimal balanced growth rate of the economy and the share of \(G\) in GDP \((\beta)\) is always negative.*

**Proof:** From (13’) \[
\frac{\partial \gamma_c}{\partial \beta} = \frac{1}{\theta} \frac{\beta^{\frac{\beta}{1-\beta}}}{(1-\beta)} (\ln \beta) < 0, \text{ since } \beta \in (0;1). \quad \blacksquare
\]

\(^{26}\) In the version of the Barro (1990)’s model proposed by Barro and Sala-i-Martin (2004, p. 221, Eq. 4.39, with \(A = L = 1\)). \(\alpha\) and \((1-\alpha)\) denote, respectively, the shares of \(K\) and the share of \(G\) in GDP.

\(^{27}\) Eq. (4.43) in Barro and Sala-i-Martin (2004, p. 222).
Unlike Barro (1990), in our model the same relationship can also be positive, as stated by the following theorem.

**THEOREM 8:** THE EFFECT OF AN INCREASE IN $\alpha$ ON THE OPTIMAL BALANCED GROWTH RATE

In an economy described by Eqs. (2) and (3), and in which:
- The aggregate production function is Cobb-Douglas (Eq. 1');
- The instantaneous utility function is CRRA;
- The allocation of public capital to goods-production ($s_r$) is endogenous,

an increase in $\alpha$ has a positive effect on the optimal balanced growth rate ($\gamma > 0$), provided that:

$$\theta > \max \{0; \varphi\} \quad \text{and} \quad \rho \in (0;1) \quad \text{(T8.1)}$$

$$0 < \gamma < \frac{1-\rho}{\theta-\varphi} \quad \text{(T8.2)}$$

$$\ln\left(\frac{\rho+\theta\gamma}{1-\alpha}\right) > \alpha \quad \text{(T8.3)}$$

**Proof:** Implicit differentiation of Eq. (7) with respect to $\alpha$ gives:

$$\frac{\partial \gamma}{\partial \alpha} = \frac{\Sigma}{\Lambda},$$

where:

$$\Sigma \equiv [(1-\rho)-(\theta-\varphi)\gamma] \left(\frac{1-\alpha}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha}} \frac{(\rho+\theta\gamma)^{\frac{1-\alpha}{\alpha}}}{\alpha} \left[ \frac{1}{\alpha} \ln\left(\frac{1}{1-\alpha}\right) - 1 + \frac{\ln(\rho+\theta\gamma)}{\alpha} \right],$$

and

$$\Lambda \equiv \alpha(\theta-\varphi) \left(\frac{1-\alpha}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha}} \frac{(\rho+\theta\gamma)^{\frac{1-\alpha}{\alpha}}}{\alpha} \left(\frac{\theta}{\alpha}\right).$$

With $\gamma > 0$ and $\theta > 0$, $\Lambda$ is always positive as long as $\theta > \varphi$. For $\Sigma$ to be always positive, instead, we require (sufficient conditions):

$$(1-\rho)-(\theta-\varphi)\gamma > 0$$

and

$$\frac{1}{\alpha} \ln\left(\frac{1}{1-\alpha}\right) - 1 + \frac{\ln(\rho+\theta\gamma)}{\alpha} > 0. \quad \blacksquare$$

Inequalities (T8.1) are always satisfied under usual values dictated by empirical evidence (for instance, $\theta \geq 1$ and $\rho$ between 0 - 2%). Restriction (T8.2) requires that the positive, optimal, balanced growth rate ($\gamma$) of the economy be bounded from above. Both empirical and historical evidence suggests that examples of unbounded growth are not known. Thus, constraint (T8.3) is probably the most relevant in guaranteeing $\partial \gamma / \partial \alpha > 0$. It is evident that the larger $\alpha$ (i.e., the share of public capital in GDP) and $\rho + \theta \gamma$ (i.e., the social time-preference rate), and the more likely it is that constraint (T8.3) is ultimately satisfied. In other words, for given $\rho$, $\theta$, $\varphi$ and $\gamma$ satisfying (T8.1) and (T8.2), (T8.3) reveals that a further rise of $\alpha$ exhibits a positive effect on economic growth provided that the share of public capital in GDP is larger than a given threshold.

Thus, treating $G$ as a stock variable – whose allocation between production of goods and provision of new public capital is endogenous – would allow a benevolent social planner observing important non-linearities in the long run relationship between optimal growth and the GDP-share of public capital (spending). This outcome is consistent with the evidence that government productive
expenditure on infrastructure-capital is subject to “critical mass” or “network” effects (Pushak et al., 2007 and Kellenberg, 2009). In this respect, our analysis clearly points out that a rise of the share of this expenditure in GDP would yield a positive impact on growth if a certain lower bound has already been exceeded.

7. Concluding Remarks

This paper has extended the productive government expenditure–augmented endogenous growth model (Barro, 1990) along two different lines. First, we viewed government expenditure as a stock (public capital), rather than a flow (general governmental purchases of goods and services). Secondly, in addition to the usual interplay between private and public capital as inputs in the same aggregate production function for consumption goods, we allowed for another possible link between the two types of capital: public and private capital-investments have been considered in our setting as two activities that, conditional on the sign of a key parameter, may be either complementary or substitutes for each other. Thus, and as a whole, we allowed for the presence of two different possible sources of interaction between the two reproducible capital inputs: the first has to do with their stocks and takes place within the final output sector through the specification of the aggregate production function (Cobb-Douglas vs. CES), while the second involves their rates of investment. The last key-ingredient of the model was that each period productive public capital can be used in part as an input in final output production and in part to accumulate furthermore the existing stock of public capital. Under the assumption that at any time the stock of public capital can always be clearly split between its two alternative uses, we have analyzed separately the cases of a constrained and unconstrained public sector (in the first case the allocation of public capital to final output-production is exogenously given, whereas in the second case it is chosen in accordance with the solution to an intertemporal optimization problem). Through this modeling strategy we were able to answer the following three questions: What is the effect that a change in the degree of complementarity/substitutability between private and public capital investments can have on optimal growth? What is the impact that an increase in the extent at which public and private capital are substitutes/complementary for each other in final output production can yield on the optimal growth rate of the economy? How can results concerning the link between optimal growth and the share of government expenditure in GDP change in comparison with the standard Barro (1990)’s model?

Our analysis reveals that when the allocation of public capital to the final output sector is exogenous, irrespective of the form of the aggregate technology for goods production a higher complementarity between private and public capital investments is unambiguously conducive to higher growth rates. On the other hand, with endogenous allocation of public capital to final output-production this continues to be true provided that the inverse of the intertemporal elasticity of substitution in consumption is sufficiently large. We showed that this condition is always satisfied under logarithmic instantaneous preferences. The relationship between optimal growth and the degree of substitutability/complementarity between the two forms of capital in the consumption goods sector is a priori not obvious. At this aim, we provided conditions for this relation to be either always positive, or always negative. Finally, we found that, unlike Barro (1990), the relationship between optimal growth and the share of productive government expenditure in GDP can be positive and is characterized by threshold-effects. While this result is in line with empirical evidence, its most important policy implication is that in the long-run a further increase in the government expenditure/GDP ratio can be associated to higher economic growth if a certain lower bound for this ratio has already been reached.

For future research we believe that a thorough econometric analysis of the main theoretical predictions of this paper would certainly contribute to shed a new light on the role of productive government services viewed as a stock in long-run optimal economic growth.
REFERENCES


**NOTES TO THE REFEREES / TECHNICAL APPENDICES NOT TO BE PUBLISHED**

**APPENDIX A:** The Social Planner’s Intertemporal Optimization Problem with Cobb-Douglas/CES production function and exogenous $s_y \in (0;1)$.

With Cobb-Douglas aggregate production function, the Hamiltonian function $(J_t)$ associated to the social planner’s problem is:

\[
J_t = \left(\frac{C_t^{1-\theta}-1}{1-\theta}\right)e^{-\rho t} + \lambda_{K_t} \left[ \left(s_y G_t\right)^{\alpha} K_t^{1-\alpha} - C_t - (1-s_y)G_t - \phi \gamma_{K_t} G_t \right] + \lambda_{G_t} \left[ (1-s_y)G_t + \phi \gamma_{K_t} G_t \right].
\]

The social planner chooses the optimal path for consumption $(C_t)$ and takes $\gamma_{K_t}$ (the growth rate of private physical capital) as given at any time $t \geq 0$. The necessary first order conditions (FOCs) are:

\[
\begin{align*}
\frac{\partial J_t}{\partial C_t} &= 0 \quad \iff \quad e^{-\rho t} C_t^\alpha = \lambda_{K_t} \\
\frac{\partial J_t}{\partial K_t} &= -\dot{\lambda}_{K_t} \quad \iff \quad \lambda_{K_t} \left[ (1-\alpha) \left(s_y G_t\right)^{\alpha} K_t^{1-\alpha} \right] = -\dot{\lambda}_{K_t} \\
\frac{\partial J_t}{\partial G_t} &= -\dot{\lambda}_{G_t} \quad \iff \quad \lambda_{K_t} \left[ \alpha \left(s_y G_t\right)^{\alpha-1} s_y K_t^{1-\alpha} - (1-s_y) - \phi \gamma_{K_t} \right] + \lambda_{G_t} \left[ (1-s_y) + \phi \gamma_{K_t} \right] = -\dot{\lambda}_{G_t},
\end{align*}
\]

along with the two transversality conditions:

\[
\begin{align*}
\lim_{t \to \infty} \lambda_{K_t} K_t &= 0 \\
\lim_{t \to \infty} \lambda_{G_t} G_t &= 0,
\end{align*}
\]

and the given initial conditions:

$K(0) > 0$, $G(0) > 0$.

From (A3):

\[
\frac{\dot{\lambda}_{K_t}}{\lambda_{K_t}} = (1-\alpha) \left(\frac{s_y G_t}{K_t}\right)^\alpha.
\]

From (A4):

\[
\frac{\dot{\lambda}_{G_t}}{\lambda_{G_t}} = \frac{\lambda_{K_t}}{\lambda_{G_t}} \left[ \alpha \left(s_y G_t\right)^{\alpha-1} s_y K_t^{1-\alpha} - (1-s_y) - \phi \gamma_{K_t} \right] + \left[ (1-s_y) + \phi \gamma_{K_t} \right].
\]

Along the BGP all time-dependent variables grow at constant (possibly positive) exponential rates and the ratio of the two endogenous state-variables, $K_t / G_t$, remains invariant over time (see the
BGP definition given in the main text). Using (A8), this definition implies that the ratio \( \frac{\lambda_{Kt}}{\lambda_{Gt}} \) is constant along the BGP equilibrium, that is:

\[
\frac{\dot{\lambda}_{Kt}}{\lambda_{Kt}} = \frac{\dot{\lambda}_{Gt}}{\lambda_{Gt}}.
\]

Equating (A7) and (A8) leads to:

\[
(A9) \quad (1-\alpha) \left( \frac{s_y G_t}{K_t} \right)^{\alpha} = \frac{\lambda_{Kt}}{\lambda_{Gt}} \left( \alpha s_y \left( \frac{K_t}{s_y G_t} \right)^{1-\alpha} \right) + \left( 1-s_y \right) + \phi \gamma_K \left( 1-\frac{\lambda_{Kt}}{\lambda_{Gt}} \right).
\]

From the dynamic constraint for \( K_t \):

\[\dot{K}_t = (s_y G_t)^{\alpha} K_t^{1-\alpha} - C_t - \dot{G}_t,\]

we obtain:

\[
(A10) \quad \gamma_K \equiv \frac{\dot{K}_t}{K_t} = s_y \left( \frac{G_t}{K_t} \right)^{\alpha} - \frac{C_t}{K_t} \left( \frac{G_t}{G_t} \right) \frac{G_t}{K_t}.
\]

Plugging (A10) into (A9) yields:

\[
(A9') \quad (1-\alpha) \left( \frac{s_y G_t}{K_t} \right)^{\alpha} = \frac{\lambda_{Kt}}{\lambda_{Gt}} \left( \alpha s_y \left( \frac{K_t}{s_y G_t} \right)^{1-\alpha} \right) + \left( 1-\frac{\lambda_{Kt}}{\lambda_{Gt}} \right) \left( 1-s_y \right) + \phi \left( s_y \left( \frac{G_t}{K_t} \right)^{\alpha} - \frac{C_t}{K_t} - \frac{G_t}{G_t} \right).
\]

With \( \frac{\lambda_{Kt}}{\lambda_{Gt}} \), \( \frac{G_t}{G_t} \) and \( \frac{G_t}{K_t} \) constant and \( s_y \) exogenous and constant, (A9’) suggests that along the BGP equilibrium \( \frac{C_t}{K_t} \) must also be constant. Therefore, along the BGP equilibrium we have:

\[
\frac{\dot{G}_t}{G_t} = \frac{\dot{K}_t}{K_t} = \frac{\dot{C}_t}{C_t} \equiv \gamma.
\]

Using the equation above together with the aggregate production function (Eq. 1’ in the main text), it follows:

\[
\frac{\dot{G}_t}{G_t} = \frac{\dot{K}_t}{K_t} = \frac{\dot{C}_t}{C_t} = \frac{\dot{Y}_t}{Y_t} \equiv \gamma.
\]

Combination of (A2) and (A7) leads to:

\[
(A11) \quad \frac{\dot{C}_t}{C_t} \equiv \gamma = \frac{1}{\theta} \left[ (1-\alpha) s_y \left( \frac{G_t}{K_t} \right)^{\alpha} - \rho \right].
\]

Moreover, from (2) in the main text:

\[
(A12) \quad \gamma \equiv \frac{\dot{G}_t}{G_t} = (1-s_y) + \phi \gamma_K = (1-s_y) + \phi \gamma
\]

Resolution of (A12) in \( \gamma \) delivers the common growth rate of variables along the BGP equilibrium:

\[
(A13) \quad \gamma = \frac{1-s_y}{1-\phi}.
\]

Substituting for \( \gamma \) in (A11) and solving that equation in \( G_t / K_t \) gives:
Given the expressions for $G/K$ and $\gamma$, from (A10) it is also possible to obtain the optimal BGP equilibrium value of the ratio $C/K$. Finally, using the aggregate production function and (A14) allows us to obtain an expression for the ratio $G/Y$ along the BGP equilibrium:

\[
\frac{G_t}{Y_t} = \frac{1}{s_y} \left[ \frac{\theta \left(1 - s_y \right)}{1 - \varphi} + \rho \right] \left\lfloor \frac{\theta}{1 - \varphi} \right\rfloor^{1/\alpha}.
\]

We now want to find a simple (sufficient) restriction for the two transversality conditions to hold along the BGP equilibrium. From the first transversality condition:

\[
\lim_{t \to \infty} \lambda_{Kt} K_t = \lim_{t \to \infty} \lambda_{K0} e^{\lambda_{K0}} K(0) e^{-\gamma t} = \lambda_{K0} K(0) \lim_{t \to \infty} e^{-\gamma t} = 0,
\]

where $\lambda_{K0}$ and $K(0)$ represent, respectively, the initial values (at $t = 0$) of the co-state variable $\lambda_K$ and of the stock of private capital $K$. By combining (A7), (A14) and (A13) the equation above can be recast as:

\[
\lambda_{K0} K(0) \lim_{t \to \infty} e^{-\gamma t} = \lambda_{K0} K(0) \lim_{t \to \infty} e^{-\gamma \theta t} = 0.
\]

For given $\lambda_{K0} = \frac{1}{c(0)^\varphi} > 0$ (see A2) and $K(0) > 0$, $\gamma > 0$, $\rho > 0$, this condition is certainly checked for each $\theta \geq 1$ (see the discussion about the existing point-estimates of $\theta$ presented in the main text). Along the BGP equilibrium: $\frac{\dot{K}_t}{K_t} = \frac{\dot{G}_t}{G_t}$ and $\frac{\dot{G}_t}{G_t} = \frac{\dot{K}_t}{K_t} = \gamma$. Hence, from the second transversality condition we similarly have:

\[
\lim_{t \to \infty} \lambda_{Gt} G_t = \lim_{t \to \infty} \lambda_{G0} e^{\lambda_{G0}} G(0) e^{-\gamma t} = \lambda_{G0} G(0) \lim_{t \to \infty} e^{-\gamma t} = \lambda_{G0} G(0) \lim_{t \to \infty} e^{-\gamma \theta t} = 0,
\]

where $\lambda_{G0}$ and $G(0)$ represent, respectively, the initial values (at $t = 0$) of the co-state variable $\lambda_G$ and of the stock of public capital $G$. Again, for given $\lambda_{G0} > 0$, $G(0) > 0$, $\gamma > 0$ and $\rho > 0$, this condition is clearly checked for each $\theta \geq 1$.

Thus far a Cobb-Douglas aggregate production function in the goods sector (i.e., $Y_t = K_t^{1-\alpha} G_t^\alpha$) has been used. Consider now the case in which the aggregate technology is CES:

\[
Y_t = \left[ (1 - \alpha) K_t^{\varphi} + \alpha G_t^\varphi \right]^{\gamma/\varphi}.
\]

In this case, the Hamiltonian function $(J_t)$ associated to the social planner’s problem would be:
\[(A1') \quad J_t = \left(\frac{C_t^{\gamma - \theta} - 1}{1 - \theta}\right) e^{-\rho t} + \lambda_{Kt}\left\{ (1 - \alpha) K_t^\alpha + \alpha (s_t G_t) \right\}^{1/\varepsilon} - C_t - (1 - s_y) G_t - \varphi \gamma_{Kt} G_t + \lambda_{Gt} \left[ (1 - s_y) G_t + \varphi \gamma_{Kt} G_t \right].\]

By using exactly the same procedure we have followed above, it is possible to show that the next results do hold along a BGP equilibrium defined as before:

\[(A13') \quad \frac{\dot{G}_t}{G_t} = \frac{\dot{K}_t}{K_t} = \frac{\dot{C}_t}{C_t} = \frac{\dot{Y}_t}{Y_t} \equiv \gamma = \frac{1 - s_y}{1 - \varphi}.\]

\[(A14') \quad \frac{G_t}{K_t} = \frac{1}{s_t \alpha^\varepsilon} \left\{ \theta \left(1 - s_y\right) + \rho \left(\frac{1 - \varepsilon}{1 - \varphi} \right) \right\}^{\frac{1 - \varepsilon}{\varepsilon}} \left(1 - \alpha\right)^{\frac{1}{\varepsilon}}.\]

\[(A15') \quad \frac{G_t}{Y_t} = \frac{1}{(1 - \alpha) \left(\frac{G_t}{K_t}\right)^{-\varepsilon} + \alpha s_t^\varepsilon} \left(1 - \alpha\right)^{\frac{1}{\varepsilon}}, \quad \text{where} \quad \frac{G_t}{K_t} \text{ is provided by the previous Eq. (A14').}\]

As it is well-known, with CES–technology when \(\varepsilon\) approaches zero the elasticity of substitution between public and private capital in the production of final output approaches one and the CES–technology becomes Cobb-Douglas. In this case \((\varepsilon \to 0)\) it is possible to show numerically\(^{28}\) that Eqs. (A13'), (A14') and (A15') collapse into their respective counterparts (Eqs. A13, A14 and A15) for each admissible value of \(\alpha, s_y, \rho, \theta\) and \(\varphi\).

**APPENDIX B:** *The Social Planner’s Intertemporal Optimization Problem with Cobb-Douglas production function and endogenous \(s_y\).*

The Hamiltonian function related to problem (SP2) in the main text is:

\[(B1) \quad J_t = \left(\frac{C_t^{\gamma - \theta} - 1}{1 - \theta}\right) e^{-\rho t} + \lambda_{Kt}\left\{ (s_t G_t)^\alpha K_t^{1 - \alpha} - C_t - (1 - s_y) G_t - \varphi \gamma_{Kt} G_t \right\} + \lambda_{Gt} \left[ (1 - s_y) G_t + \varphi \gamma_{Kt} G_t \right].\]

The social planner now chooses the optimal paths for consumption \((C)\) and the sectoral distribution of public capital between production of goods and provision of new public capital \((s_y)\). As before, s/he takes \(\gamma_k\) (the growth rate of private capital) as given at any time \(t \geq 0\). The necessary FOCs read as:

\[(B2) \quad \frac{\partial J_t}{\partial C_t} = 0 \iff \frac{e^{-\rho t}}{C_t^\theta} = \lambda_{Kt};\]

\[(B3) \quad \frac{\partial J_t}{\partial s_{yt}} = 0 \iff \lambda_{Kt} \left[ \alpha \left(\frac{K_t}{s_{yt} G_t}\right)^{1 - \alpha} + 1 \right] = \lambda_{Gt};\]

\[(B4) \quad \frac{\partial J_t}{\partial K_t} = -\dot{\lambda}_{Kt} \iff \lambda_{Kt} \left(1 - \alpha\right) \left(\frac{s_{yt} G_t}{K_t}\right)^\alpha = -\dot{\lambda}_{Kt};\]

---

\(^{28}\) For instance, by using Mathematica for Microsoft Windows.
\[(B5) \quad \frac{\partial J_i}{\partial G_i} = -\dot{\lambda}_{Gi} \iff \lambda_{Ki} \left[ \alpha s_{yi} \left( \frac{K_i}{s_{yi} G_i} \right)^{1-\alpha} - (1-s_{yi}) - \varphi \gamma_{Ki} \right] + \dot{\lambda}_{Gi} \left[ (1-s_{yi}) + \varphi \gamma_{Ki} \right] = -\dot{\lambda}_{Gi} , \]

along with the two transversality conditions:
\[
\lim_{t \to \infty} \lambda_{Ki} K_i = 0
\]
\[
\lim_{t \to \infty} \lambda_{Gi} G_i = 0 ,
\]
and the given initial conditions:
\[
K(0) > 0, \quad G(0) > 0 .
\]

Using (6) and (4’) in the main text, Eqs. (B3), (B4) and (B5) can be re-written as:

(B3’)
\[
\lambda_{Ki} (1 + P_{Gi}) = \lambda_{Gi}
\]

(B4’)
\[
\lambda_{Ki} r_i = -\dot{\lambda}_{Ki}
\]

(B5’)
\[
\lambda_{Ki} \left[ s_{yi} P_{Gi} - (1-s_{yi}) - \varphi \gamma_{Ki} \right] + \dot{\lambda}_{Gi} \left[ (1-s_{yi}) + \varphi \gamma_{Ki} \right] = -\dot{\lambda}_{Gi} .
\]

Along the BGP equilibrium:
\(i\) All time-dependent variables grow at constant (possibly positive) exponential rates;
\(ii\) The ratio of the two endogenous state variables, \( K / G \), remains invariant over time.

From (i) it follows that along the BGP \( \frac{\dot{G}_i}{\dot{G}_i} = \gamma \) and \( \frac{\dot{K}_i}{\dot{K}_i} = \gamma_k \) are constant. Therefore, \( s_{yi} \) must be also constant along the BGP. From (ii), instead, it follows that:

(B6)
\[
\gamma = \gamma_k = \gamma .
\]

Using (B6) into the dynamic constraint for \( G \) we get:

(B7)
\[
\gamma = \gamma_k = \gamma = \frac{1-s_{yi}}{1-\varphi} .
\]

Combining equations (B3’) and (B5’) yields:

(B8)
\[
\lambda_{Ki} P_{Gi} (1 + \varphi \gamma) = -\dot{\lambda}_{Gi} .
\]

Using (B3’) into (B8) leads to:

(B8’)
\[
\frac{\dot{\lambda}_{Gi}}{\lambda_{Gi}} = \frac{P_{Gi}}{1 + P_{Gi}} (1 + \varphi \gamma) .
\]

By using (B4’):

(B9)
\[
-\frac{\dot{\lambda}_{Ki}}{\lambda_{Ki}} = r_i.
\]

Along the BGP equilibrium \( P_{Gi} \) is constant (see Eq. 4’ in the main text and the definition of BGP equilibrium). Therefore, from (B3’):

(B3’’)
\[
\frac{\dot{\lambda}_{Ki}}{\lambda_{Ki}} = \frac{\dot{\lambda}_{Gi}}{\lambda_{Gi}} .
\]

Plugging (B8’) and (B9) into (B3’’) leads to:

(B10)
\[
r_i = \frac{P_{Gi}}{1 + P_{Gi}} (1 + \varphi \gamma) = r .
\]

Eq. (B10) says that along the BGP equilibrium \( r \) is constant, as well (see also Eq. 6 in the main text and the definition of BGP equilibrium). Using (6) and (4’) in the main text into (B10) gives:
\[(B11) \quad \frac{s_y G_i}{K_i} = \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{s_y G_i}{K_i} \right)^{1-\alpha} (1+\varphi \gamma) \left( \frac{s_y G_i}{K_i} \right)^{-1} + \alpha \].

In the presence of an interior solution for \( s_y \) – that is \( s_y (0;1) \) – and with \( G_i / K_i > 0 \) the left hand side of (B11) is unambiguously positive. Therefore, the right hand side of the same equation should be also positive. Since \( \alpha \in (0;1) \), this is true when:

\[(\text{CONSTRAINT 1}) \quad (1+\varphi \gamma) > 0\].

In a moment we shall show that the solution to the social planner’s problem satisfies this constraint.

With both sides of (B11) being positive, we can raise them to power \( \alpha \) and obtain:

\[(B11') \quad \left( \frac{s_y G_i}{K_i} \right) = \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{s_y G_i}{K_i} \right)^{a(1-\alpha)} (1+\varphi \gamma)^a \left( \frac{s_y G_i}{K_i} \right)^{-1} + \alpha \].

From the two dynamic constraints (in \( K \) and \( G \)) jointly considered, on the one hand, and (B7), on the other, it follows:

\[\gamma = \left( \frac{s_y G_i}{K_i} \right)^{-\alpha} - \frac{C_i}{K_i} - \frac{(1-s_y)}{1-\varphi} \frac{G_i}{K_i} - \varphi \gamma \frac{G_i}{K_i} \].

Along the BGP equilibrium, \( \gamma, \ s_y \) and \( K / G \) are constant. Hence, \( C / K \) will be constant as well. This implies:

\[(B7') \quad \gamma_c = \gamma_k = \gamma_c \equiv \gamma = \frac{1-s_y}{1-\varphi}, \quad \text{where} \quad \gamma_c \equiv \gamma, \quad \gamma_c \quad \text{is the growth rate of consumption.}\]

After combining (B2), (B9) and (B7') we obtain:

\[r = \rho + \theta \gamma,\]

and (after using Eq. 6 in the main text):

\[(B12) \quad \left( \frac{s_y G_i}{K_i} \right) = \left( \frac{\rho + \theta \gamma}{1-\alpha} \right).\]

Note that (B11') and (B12) represent together relations between two of the endogenous variables of the model (respectively, \( s_y G_i / K_i \) and \( \gamma \)) and its parameters (\( \alpha, \varphi, \rho, \theta \)). So, we have two equations (B11' and B12) in two unknowns (\( s_y G_i / K_i \) and \( \gamma \)). We focus on positive and constant optimal balanced growth rates (\( \gamma > 0 \)). After equating (B11') and (B12), we end up with:

\[(B12') \quad \left( \frac{s_y G_i}{K_i} \right)^{1-\alpha} = \alpha \left( \frac{\rho + \theta \gamma}{1-\alpha} \right)^{1/\alpha} \left( \frac{\alpha}{1-\alpha} \right)^{1/\alpha} - \left( \frac{\rho + \theta \gamma}{1-\alpha} \right)^{1/\alpha}.\]

The left hand side of this expression is positive for each \( s_y \in (0;1) \) and \( G / K > 0 \). Therefore, under the same economically-meaningful restrictions, the right hand side should be also positive. With \( \alpha \in (0;1) \) and \( \left( \frac{\rho + \theta \gamma}{1-\alpha} \right) = \left( \frac{r}{1-\alpha} \right) > 0 \), this implies:
In a moment we shall show that the solution to the social planner’s problem satisfies this constraint, as well. With Constraint 2 checked, from (B12’) it easily follows:

\begin{equation} \alpha (\rho + \theta \gamma ) + \frac{1}{1-\alpha} (\rho + \theta \gamma )^{\frac{1}{1-\alpha}} = \alpha (1 + \varphi \gamma ). \end{equation}

In principle, solving (B14) in terms of \( \gamma \) would amount to obtain the common, constant and positive optimal balanced growth rate of this economy. Unfortunately, (B14) cannot be solved in closed form in terms of \( \gamma \) (unless one makes specific assumptions on \( \alpha \) and, thus, on the form of the aggregate production function). However, we in general observe that the economy’s optimal balanced growth rate depends implicitly on the preference (\( \rho \) and \( \theta \)) and the technological (\( \alpha \) and \( \phi \)) parameters of the model. Given \( \gamma(\alpha, \varphi, \rho, \theta) \), implicitly determined by (B14),

- Eq. (B7) allows determining univocally the optimal \( s_y(\alpha, \varphi, \rho, \theta) \);
- Given \( \gamma(\alpha, \varphi, \rho, \theta) \) and \( s_y(\alpha, \varphi, \rho, \theta) \), (B13) determines univocally the optimal \( \frac{G_i}{K_i}(\alpha, \varphi, \rho, \theta) \);
- Finally, using the aggregate production function, and given the optimal \( s_y(\alpha, \varphi, \rho, \theta) \) and \( \frac{G_i}{K_i}(\alpha, \varphi, \rho, \theta) \), it is also possible to obtain the optimal value of \( \frac{G_i}{Y_i}(\alpha, \varphi, \rho, \theta) \).

We now show that the solution to the social planner’s problem (Eq. B14) satisfies CONSTRAINTS 1 and 2 written above. Indeed, with \( \gamma > 0 \):

- The left hand side of (B14) is strictly positive, implying that it should be \( (1 + \varphi \gamma) > 0 \). This is CONSTRAINT 1;
- Algebraic manipulation of (B14) also implies:
  \[
  \left( \frac{\alpha}{1-\alpha} \right) (1 + \varphi \gamma) = \left( \frac{\rho + \theta \gamma}{1-\alpha} \right) (\rho + \theta \gamma)^{\frac{1}{1-\alpha}} > \left( \frac{\rho + \theta \gamma}{1-\alpha} \right)^{\frac{1}{1-\alpha}} ,
  \]
  which represents CONSTRAINT 2.

At the end of this Appendix, we want to find a simple (sufficient) restriction for the two transversality conditions to hold along the BGP equilibrium. From the first transversality condition:

\[
\lim_{t \to 0^+} \lambda_K = \lim_{t \to 0^+} \lambda_{K0} e^{\frac{\kappa_0}{\kappa}} K(0) e^{\frac{\kappa}{\kappa_0}} = \lambda_{K0} K(0) \lim_{t \to \infty} e^{\frac{\kappa_0}{\kappa} \lambda} = 0,
\]
where $\lambda_{k0} = \frac{1}{C(0)^{\theta}} > 0$ (see B2) and $K(0) > 0$ represent, respectively, the initial values (at $t = 0$) of the co-state variable $\lambda_k$ and of the stock of private capital $K$. By using (B9), (B7’) and the fact that $r = \rho + \theta \gamma$, in the end the equation above can be recast as:

$$\lambda_{k0} K(0) \lim_{t \to \infty} e^{\left[ \frac{\lambda_{k0}}{K(0)} - \frac{\lambda_{k0}}{\lambda_k} \right]} = \lambda_{k0} K(0) \lim_{t \to \infty} e^{-[\gamma(\theta - 1) + \rho]t} = 0.$$  

For given $\lambda_{k0} > 0$, $K(0) > 0$, $\rho > 0$ and with $\gamma > 0$, this condition is certainly checked for each $\theta \geq 1$. Along the BGP equilibrium: $\frac{\dot{K}}{K} = \frac{\dot{G}}{G}$ (Eq. B3”) and $G_t = \frac{K}{K} = \gamma$ (Eq. B7). Hence, from the second transversality condition we similarly have:

$$\lim_{t \to \infty} \lambda_{g0} G_t = \lim_{t \to \infty} \lambda_{g0} e^{\left[ \frac{\dot{\lambda}_{g0}}{G(0)e} \right]_{t}} = \lambda_{g0} G(0) \lim_{t \to \infty} e^{-[\gamma(\theta - 1) + \rho]t} = 0,$$

where $\lambda_{g0}$ and $G(0)$ represent, respectively, the initial values (at $t = 0$) of the co-state variable $\lambda_g$ and of the stock of public capital $G$. In the last expression, Eq. (B9) and $r = \rho + \theta \gamma$ have been used. Again, for given $\lambda_{g0} > 0$, $G(0) > 0$, $\rho > 0$ and with $\gamma > 0$, this condition is clearly checked for each $\theta \geq 1$. ■

**APPENDIX C: The Social Planner’s Intertemporal Optimization Problem with CES–production function and endogenous $s_y$**

The Hamiltonian function related to this problem is:

$$(C1) \quad J_t = \left[ \frac{C^\theta}{1-\theta} - 1 \right] e^{-r t} + \lambda_{k_t} \left[ \left(1 - \alpha\right) K_t^\epsilon + \alpha \left( \frac{s_y G_t}{K_t} \right)^\epsilon \right]^{\frac{1}{1-\epsilon}} - C_t - \left(1 - s_y\right) G_t - \phi \gamma K_t G_t + \lambda_{g_t} \left[ \left(1 - s_y\right) G_t + \phi \gamma K_t G_t \right].$$

The social planner chooses the optimal paths for consumption ($C$) and the sectoral distribution of public capital between production of consumption goods and provision of new public infrastructure-capital ($s_y$) by taking $\gamma_k$ (the growth rate of private capital) as given at any time $t \geq 0$. The necessary FOCs read as:

$$(C2) \quad \frac{\partial J_t}{\partial C_t} = 0 \iff e^{-r t} C_t^{\theta} = \lambda_{k_t};$$

$$(C3) \quad \frac{\partial J_t}{\partial s_y} = 0 \iff \lambda_{k_t} \left[ \alpha \left(1 - \alpha\right) \left( \frac{K_t}{s_y G_t} \right)^\epsilon \right]^{\frac{1}{1-\epsilon}} + 1 = \lambda_{g_t};$$

$$(C4) \quad \frac{\partial J_t}{\partial K_t} = -\dot{\lambda}_{k_t} \iff \lambda_{k_t} \left(1 - \alpha\right) \left[ \alpha \left(1 - \alpha\right) \left( \frac{s_y G_t}{K_t} \right)^\epsilon \right]^{\frac{1}{1-\epsilon}} = -\dot{\lambda}_{k_t}.$$
(C5) \[ \frac{\partial J_t}{\partial G_t} = -\dot{\lambda}_{Gt} \iff \dot{\lambda}_{Kt} \left\{ \alpha s_Y \left( 1 - \alpha \right) \left( \frac{K_t}{s_Y G_t} \right)^{\epsilon} + \alpha \right\}^{\frac{1-\epsilon}{\epsilon}} - (1 - s_Y) - \varphi \gamma_{Kt} + \lambda_{Gt} \left[ (1 - s_Y) + \varphi \gamma_{Kt} \right] = -\dot{\lambda}_{Gt}, \]

along with the two transversality conditions:

\[ \lim_{t \to \infty} \dot{\lambda}_{Kt} K_t = 0 \]
\[ \lim_{t \to \infty} \dot{\lambda}_{Gt} G_t = 0, \]

and the given initial conditions:

\[ K(0) > 0, \quad G(0) > 0. \]

As the CES production function exhibits constant returns to scale to \( K \) and \( G \) together and the sector producing final goods is perfectly competitive, at equilibrium the two factor-inputs (\( K \) and \( G \)) receive their own marginal productivities (in terms of final output, \( Y \)). These productivities are written in the main text (Eqs. 4” and 6’). Using these two equations, (C3), (C4) and (C5) can be recast as:

(C3’) \[ \lambda_{Kt} (1 + P_{Gr}) = \lambda_{Gt}; \]

(C4’) \[ \lambda_{Kt} r_t = -\dot{\lambda}_{Kt}; \]

(C5’) \[ \lambda_{Gt} \left[ s_W P_{Gr} - (1 - s_Y) - \varphi \gamma_{Kr} \right] + \lambda_{Gt} \left[ (1 - s_Y) + \varphi \gamma_{Kr} \right] = -\dot{\lambda}_{Gt}. \]

Along the BGP equilibrium: (i) All time-dependent variables grow at constant (possibly positive) exponential rates; (ii) The ratio of the two endogenous state variables, \( K_t / G_t \), remains invariant over time. From (i) it follows that, along the BGP, \( \dot{G}_t / G_t \equiv \gamma_G \) and \( \dot{K}_t / K_t \equiv \gamma_K \) are constant. Therefore, \( s_Y \) must be also constant along the BGP (see the dynamic constraint for \( G \)). From (ii), instead, it follows that:

(C6) \[ \gamma_G = \gamma_K \equiv \gamma. \]

Using (C6) into the dynamic constraint for \( G \), we get:

(C7) \[ \gamma_G = \gamma_K \equiv \gamma = \frac{1 - s_Y}{1 - \varphi}. \]

Combining equations (C3’) and (C5’) yields:

(C8) \[ \lambda_{Kt} P_{Gr} (1 + \varphi \gamma) = -\dot{\lambda}_{Gt}. \]

Using (C3’) into (C8) leads to:

(C8’) \[ -\frac{\dot{\lambda}_{Gt}}{\lambda_{Gt}} = \frac{P_{Gr}}{1 + P_{Gr}} (1 + \varphi \gamma). \]

By using (C4’):

(C9) \[ -\frac{\dot{\lambda}_{Kt}}{\lambda_{Kt}} = r_t. \]

Along the BGP equilibrium \( P_{Gr} \) is constant (see Eq. 4’” in the main text and the definition of BGP equilibrium). Therefore, from (C3’):

(C3”’) \[ \frac{\dot{\lambda}_{Kt}}{\lambda_{Kt}} = \frac{\dot{\lambda}_{Gt}}{\lambda_{Gt}}. \]

Plugging (C8’) and (C9) into (C3”’) leads to:
Eq. (C10) says that along the BGP equilibrium $r$ is also constant (see Eq. 6’ in the main text and the definition of BGP equilibrium). In what follows we focus on interior solutions – that is $s_y \in (0;1)$ – with a positive ratio of public to private capital stocks $\frac{G_t}{K_t} > 0$. Therefore, $P_o$ is always positive. Plugging (6’) and (4”) into (C10) yields:

\[
(C11) \left(\frac{s_y G_t}{K_t}\right)^{\frac{1}{\epsilon}} = \left(\frac{1 + \varphi \gamma}{1 - \alpha}\right) - \left(1 - \alpha\right) + \alpha \left(\frac{s_y G_t}{K_t}\right)^{\frac{1}{\epsilon}}.
\]

With $\epsilon \in (-\infty; 0) \cup (0; 1)$ the left hand side of (C11) is always positive. Therefore, the right hand side must be positive, too. This, in turn, implies that $(1 + \varphi \gamma)$ should be sufficiently large, that is:

\[
(1 + \varphi \gamma) > \left(1 - \alpha\right) + \alpha \left(\frac{s_y G_t}{K_t}\right)^{\frac{1}{\epsilon}} > 0.
\]

When this restriction is met, $r$ is always positive, too (see C10 above and Eq. 6’ in the main text). From the dynamic constraint for $K_t$, it follows:

\[
\gamma \left(1 + \varphi \frac{G_t}{K_t}\right) = \left(1 - \alpha\right) + \alpha \left(\frac{s_y G_t}{K_t}\right)^{\frac{1}{\epsilon}} \frac{1 - C'}{C} - \frac{1 - s_y}{1 - \varphi} \frac{G_t}{K_t}.
\]

Along the BGP equilibrium, $\gamma$, $s_y$ and $\frac{K_t}{G_t}$ are constant. Hence, $\frac{C}{K}$ will be constant as well. This implies:

\[
(C7') \quad \gamma = \gamma_c \equiv \frac{1 - s_y}{1 - \varphi}, \quad \text{where } \gamma_c \equiv \frac{C}{K} \text{ is the growth rate of consumption.}
\]

After combining (C2), (C9) and (C7’) we first obtain the usual Euler Equation:

\[
r = \rho + \theta \gamma,
\]

and then (using Eq. 6’ in the main text):

\[
(C12) \quad \left(\frac{s_y G_t}{K_t}\right)^{\frac{1}{\epsilon}} = \frac{1}{\alpha} \left[\left(\frac{\rho + \theta \gamma}{1 - \alpha}\right)^{\frac{1}{\epsilon}} - (1 - \alpha)\right].
\]

The left hand side of (C12) is always positive. Therefore, the right hand side should be positive, as well. With $\alpha \in (0; 1)$, this implies:

\[
(\text{CONSTRAINT 1}) \quad (\rho + \theta \gamma)^{\frac{1}{\epsilon}} > (1 - \alpha).
\]

When (Constraint 1) is met, the left and the right hand sides of (C12) are both positive. We can, thus, raise them first of all to power $1/\epsilon$:

\[
\left(\frac{s_y G_t}{K_t}\right)^{\frac{1}{\epsilon}} = \frac{1}{(\alpha)^{\frac{1}{\epsilon}}} \left[\left(\frac{\rho + \theta \gamma}{1 - \alpha}\right)^{\frac{1}{\epsilon}} - (1 - \alpha)\right]\frac{1}{\epsilon}
\]

and, then, to power $(1 - \epsilon)$, yielding:

\[
(C12') \quad \left(\frac{s_y G_t}{K_t}\right)^{1 - \epsilon} = \frac{1}{(\alpha)^{(1 - \epsilon)/\epsilon}} \left[\left(\frac{\rho + \theta \gamma}{1 - \alpha}\right)^{1 - \epsilon} - (1 - \alpha)\right]^{1 - \epsilon}\frac{1}{\epsilon}.
\]
Eqs. (C11) and (C12') represent relations between two of the endogenous variables of the model (s_iG_i/K_i and γ, respectively) and its parameters (ε, α, φ, ρ and θ). In other words, (C11) and (C12') together represent a system of two equations in two unknowns (s_iG_i/K_i and γ). We focus on positive and constant optimal balanced growth rates (γ > 0). After equating (C11) and (C12') we end up with:

\[
(C13) \quad \left( \frac{s_iG_i}{K_i} \right)^{\frac{\epsilon}{\alpha}} + \left( \frac{1 - \alpha}{\alpha} \right) = \frac{1}{\alpha} \left[ \frac{1 + \phi \gamma}{1 - \alpha} \right] - \frac{1}{\alpha^{\epsilon/\alpha}} \left[ \left( \frac{\rho + \theta \gamma}{1 - \alpha} \right)^{\frac{\epsilon}{1 - \alpha}} - (1 - \alpha)^{\frac{1 - \epsilon}{\epsilon}} \right].
\]

Again, with the left hand side of (C13) being definitely positive and with ε that can take also negative values, we require:

\[
(CONSTRAINT 2) \quad \left( \frac{1 + \phi \gamma}{1 - \alpha} \right) > \frac{1}{\alpha^{\epsilon/\alpha}} \left[ \left( \frac{\rho + \theta \gamma}{1 - \alpha} \right)^{\frac{\epsilon}{1 - \alpha}} - (1 - \alpha)^{\frac{1 - \epsilon}{\epsilon}} \right].
\]

In a moment we shall demonstrate that the solution to the social planner’s problem satisfies CONSTRAINTS 1 and 2. Eq. (C13) provides an expression for \( s_iG_i/K_i \) as a function of γ and the other parameters of the model (ε, α, φ, ρ and θ). Using (C13) into (C12') it is possible in the end to obtain:

\[
(C14) \quad \left[ \left( \frac{\rho + \theta \gamma}{1 - \alpha} \right)^{\frac{\epsilon}{1 - \alpha}} - (1 - \alpha)^{\frac{1 - \epsilon}{\epsilon}} \right] = \alpha^{\epsilon/\alpha} \left[ (1 - \rho) - (\theta - \phi) \gamma \right].
\]

In principle, solving (C14) in terms of γ would give the common, constant and positive optimal balanced growth rate of this economy. Unfortunately, in general Eq. (C14) cannot be solved in closed form in terms of γ. However, we see that γ depends implicitly on the preference (ρ and θ) and technological (ε, α and φ) parameters of the model.

Given γ(ε, α, φ, ρ, θ), implicitly given by (C14),

- Eq. (C7) allows determining univocally the optimal \( s_i(ε, α, φ, ρ, θ) \);
- Given γ(ε, α, φ, ρ, θ) and \( s_i(ε, α, φ, ρ, θ) \), (C13) determines univocally the optimal value of \( \frac{G_i}{K_i}(ε, α, φ, ρ, θ) \);
- Finally, using the aggregate production function and given the optimal \( s_i(ε, α, φ, ρ, θ) \) and \( \frac{G_i}{K_i}(ε, α, φ, ρ, θ) \), it is also possible to obtain the optimal value of \( \frac{G_i}{Y_i}(ε, α, φ, ρ, θ) \).

In the model we have two expressions for the rental price of private capital, \( r \) (the Euler Eq. and C10). By combining these two equations, it is possible to obtain:

\[
(C15) \quad P_g = \frac{\rho + \theta \gamma}{(1 - \rho) - (\theta - \phi) \gamma}.
\]

For \( P_g \) to be positive, again it needs to be: \( 0 < \gamma < \frac{1 - \rho}{\theta - \phi} \); \( \theta > \phi \); \( 0 < \rho < 1 \) (see Eq. 11b in the main text).

We now show that the solution to the social planner’s problem (Eq. C14) certainly satisfies CONSTRAINTS 1 and 2:
The left hand side of (C14) is always definite (for any value of \( \varepsilon \)) when \((\rho + \theta \gamma) > (1 - \alpha)\). This is CONSTRAINT 1;

Moreover, after easy algebraic manipulations, Eq. (C14) can also be written as:

\[
\frac{(\rho + \theta \gamma)^{\frac{\varepsilon}{1-\alpha}}}{1-\alpha} - (1-\alpha) = \alpha^{\frac{\varepsilon}{1-\alpha}} (1 + \varphi \gamma) - \alpha^{\frac{\varepsilon}{1-\alpha}} (\rho + \theta \gamma),
\]

which implies:

\[
\frac{(1 + \varphi \gamma)}{(1-\alpha)} = \frac{1}{\alpha^{\frac{\varepsilon}{1-\alpha}}} \left( \frac{(\rho + \theta \gamma)^{\frac{\varepsilon}{1-\alpha}}}{1-\alpha} - (1-\alpha) \right) + \frac{1}{\alpha^{\frac{\varepsilon}{1-\alpha}}} \left( \frac{(\rho + \theta \gamma)^{\frac{\varepsilon}{1-\alpha}}}{1-\alpha} - (1-\alpha) \right)^{\frac{\varepsilon}{1-\alpha}} - (1-\alpha)
\]

since \((\rho + \theta \gamma)/(1-\alpha) > 0\) for each \(\gamma > 0\). This is CONSTRAINT 2.

Finally, we want to find a simple (sufficient) restriction for the two transversality conditions to hold along the BGP equilibrium. From the first transversality condition:

\[
\lim_{t \to \infty} \lambda_{Kt} K_t = \lim_{t \to \infty} \lambda_{K0} e^{\frac{\varepsilon}{1-\alpha}} K(0) e^{\frac{\varepsilon}{1-\alpha}} = \lambda_{K0} K(0) \lim_{t \to \infty} e^{\frac{\varepsilon}{1-\alpha}} = 0,
\]

where \(\lambda_{K0} = \frac{1}{C(0)} > 0\) (see C2) and \(K(0) > 0\) represent, respectively, the initial values (at \(t = 0\)) of the co-state variable \(\lambda_k\) and of the stock of private capital \(K\). By using (C9), (C6) and the fact that \(r = \rho + \theta \gamma\) (Euler Equation), in the end the equation above can be recast as:

\[
\lambda_{K0} K(0) \lim_{t \to \infty} e^{\frac{\varepsilon}{1-\alpha}} = \lambda_{K0} K(0) \lim_{t \to \infty} e^{\frac{\varepsilon}{1-\alpha}} = 0.
\]

For given \(\lambda_{K0} > 0\), \(K(0)\), \(\rho > 0\) and \(\gamma > 0\), this condition is certainly checked for each \(\theta \geq 1\).

Along the BGP equilibrium: \(\frac{\lambda_{Gi}}{\lambda_{Ki}} = \frac{\dot{A}_{Gi}}{\lambda_{Ki}}\) (Eq. C3”) and \(\frac{G_i}{K_i} = \frac{\dot{K}_i}{\dot{K}_i} \equiv \gamma\) (Eq. C6). Hence, from the second transversality condition we similarly have:

\[
\lim_{t \to \infty} \lambda_{Gi} G_t = \lim_{t \to \infty} \lambda_{Gi} e^{\frac{\varepsilon}{1-\alpha}} G(0) e^{\frac{\varepsilon}{1-\alpha}} = \lambda_{Gi} G(0) \lim_{t \to \infty} e^{\frac{\varepsilon}{1-\alpha}} = \lambda_{Gi} G(0) \lim_{t \to \infty} e^{\frac{\varepsilon}{1-\alpha}} = 0.
\]

In the last expression Eq. (C9) and \(r = \rho + \theta \gamma\) have been used. Again, for given \(\lambda_{Gi} > 0\), \(G(0) > 0\), \(\rho > 0\) and \(\gamma > 0\), this condition is clearly checked for each \(\theta \geq 1\).

**APPENDIX D: PROOF OF THEOREM 7**

After a huge amount of algebra, implicit differentiation of Eq. (10) in the main text with respect to \(\varepsilon\) yields:

\[
(D1) \quad \frac{\partial \gamma}{\partial \varepsilon} = \eta \pi + \omega \chi,
\]
where:
\[
Z \equiv \left[ (\rho + \theta \gamma)^{\epsilon(l-1)} - (1 - \alpha)^{\epsilon(l-1)} \right]^{1-\frac{2\epsilon}{\epsilon}} (\rho + \theta \gamma)^{\epsilon(l-1)} (1 - \alpha)^{\epsilon(l-1)} (1 - \alpha)^{\epsilon(l-1)} (\theta - \varphi) \right); \\
\eta \equiv \left( \frac{1}{\epsilon} \right)^2; \\
\pi \equiv \left( (\rho + \theta \gamma)^{\epsilon(l-1)} - (1 - \alpha)^{\epsilon(l-1)} \right)^{1-\frac{2\epsilon}{\epsilon}} \ln \left[ (\rho + \theta \gamma)^{\epsilon(l-1)} - (1 - \alpha)^{\epsilon(l-1)} \right] - \alpha^{\epsilon(l-1)} (\ln \alpha)^{(1 - \rho) - (\theta - \varphi) \gamma}; \\
\omega \equiv \left( \frac{1}{1-\epsilon} \right)^{\epsilon(l-1)} - (1 - \alpha)^{\epsilon(l-1)} \right)^{1-\frac{2\epsilon}{\epsilon}}; \\
\chi \equiv \left( (1 - \alpha)^{\epsilon(l-1)} \ln (1 - \alpha) + (\rho + \theta \gamma)^{\epsilon(l-1)} \ln (\rho + \theta \gamma) \right).
\]

The term $Z$ on the left hand side of Eq. (D1) is always definite and positive as long as $(\rho + \theta \gamma)^{\epsilon(l-1)} > (1 - \alpha)$ and $\theta > \varphi$. Therefore, the sign of $\frac{\partial \gamma}{\partial \epsilon}$ depends on the sign of the right hand side of (D1). We also notice that $\eta$ is always positive and that $\omega$ is also definite and positive when $(\rho + \theta \gamma)^{\epsilon(l-1)} > (1 - \alpha)$ holds. Given all this, we easily conclude that:

- $\frac{\partial \gamma}{\partial \epsilon}$ is unambiguously positive when the following two (sufficient conditions) do hold simultaneously:
  \[
  \pi > 0 \quad \Rightarrow \quad (\rho + \theta \gamma)^{\epsilon(l-1)} - (1 - \alpha)^{\epsilon(l-1)} \quad \text{after using Eq. (10) in the main text;}
  \chi > 0 \quad \Rightarrow \quad \left( \frac{1}{\epsilon} \right)^{\epsilon(l-1)} > 0.
  \]

- $\frac{\partial \gamma}{\partial \epsilon}$ is unambiguously negative when the following two sufficient conditions do hold simultaneously:
  \[
  \pi < 0 \quad \Rightarrow \quad (\rho + \theta \gamma)^{\epsilon(l-1)} < (1 - \alpha)^{\epsilon(l-1)} \quad \text{after using Eq. (10) in the main text;}
  \chi < 0 \quad \Rightarrow \quad \left( \frac{1}{\epsilon} \right)^{\epsilon(l-1)} < 0.
  \]

In the other two possible cases – i.e., ($\pi < 0; \chi > 0$) and ($\pi > 0; \chi < 0$) – the right hand side of Eq. (D1) has ambiguous sign and, therefore, with $(\rho + \theta \gamma)^{\epsilon(l-1)} > (1 - \alpha)$ and $\theta > \varphi$, the sign of $\frac{\partial \gamma}{\partial \epsilon}$ is ambiguous, as well (it can be either positive, or negative, or else equal to zero). □