SCALAR CHARACTERIZATION OF EXPLICITLY QUASICONVEX SET-VALUED MAPS

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ABSTRACT. This paper concerns explicitly quasiconvex set-valued maps, defined on a nonempty convex subset of a real linear space with values in a partially ordered real linear space, with respect to a solid vectorially closed convex cone. It is shown that these generalized convex set-valued maps can be characterized in terms of classical explicit quasiconvexity of certain scalar functions.

1. INTRODUCTION

Real-valued explicitly quasiconvex functions (also known under different names in the literature) constitute a special class of quasiconvex functions. Enjoying certain properties similar to that of convex functions, they play an important role in classical optimization problems, as already shown in some early works (see e.g. [14]). In what concerns optimization problems with multiple criteria, several interesting questions have been solved assuming that the vector-valued objective function is componentwise explicitly quasiconvex: the connectedness and contractibility of efficient sets (see e.g. [5], [3] and [2]), the parametric methods for scalarizing bicriteria optimization problems (see e.g. [13] and [9]), and the Pareto reducibility of multicriteria optimization problems (see e.g. [10] and [11]).

In the recent paper [12] the notion of explicit quasiconvexity has been extended for set-valued maps and vector-valued functions in a more general framework, where the componentwise setting makes no sense anymore, since these functions take values in a real linear space partially ordered by an arbitrary relatively solid convex cone. The principal aim of our work is to prove that, under some mild assumptions, explicitly cone-quasiconvex set-valued maps can be characterized in terms of classical explicit quasiconvexity of certain scalar functions, in the same manner as set-valued cone-quasiconvex maps have been characterized in [4].
The paper is organized as follows. In Section 2 we state some preliminary results about solid convex cones, which allow us to adapt the notion of smallest strictly monotonic functions in the sense of Dinh The Luc [7] to our algebraic (i.e., not necessarily topological) setting. In Section 3 we state our main result (Theorem 3.7) concerning the characterization of explicitly cone-quasiconvex set-valued and we derive from it a characterization of explicitly cone-quasiconvex vector-valued functions.

2. Properties of solid convex cones

Throughout this paper $Y$ will be a linear space over the field $\mathbb{R}$ of real numbers. For convenience, let us denote $\mathbb{R}_+ := [0, +\infty[$, $\mathbb{R}_+^* := ]0, +\infty[$, and $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

Recall that the algebraic interior and the relative algebraic interior of any set $M \subset Y$ are given by:

\[
\text{cor} M := \{ x \in M \mid \forall y \in Y, \exists \lambda \in \mathbb{R}_+^*, x + [0, \lambda] \cdot y \subset M \};
\]

\[
\text{icr} M := \{ x \in M \mid \forall y \in \text{span}(M - M), \exists \lambda \in \mathbb{R}_+^*, x + [0, \lambda] \cdot y \subset M \}.
\]

The set $M$ is called solid (relatively solid) if $\text{cor} M \neq \emptyset$ (resp. $\text{icr} M \neq \emptyset$). Following [1], we say that $M$ is vectorially closed if $M = \text{vcl} M$, where the so-called vectorial closure of $M$ is given by

\[
\text{vcl} M := \{ y \in Y \mid \exists y' \in Y, \forall \lambda \in \mathbb{R}_+^*, \exists \lambda' \in ]0, \lambda], y + \lambda' y' \in M \}
\]

\[
= \{ y \in Y \mid \exists \tilde{y} \in Y, \exists \{ \lambda_n \}_{n \in \mathbb{N}} \subset \mathbb{R}, \lambda_n \rightarrow 0, y + \lambda_n \tilde{y} \in M, \forall n \in \mathbb{N} \}.
\]

As shown by Adán and Novo in [1], if $M$ is convex, then

\[
\forall t \in [0, 1[ : (1 - t) \cdot \text{icr} M + t \cdot \text{vcl} M \subset \text{icr} M.
\]

In the sequel we will assume that the real linear space $Y$ is partially ordered by a solid convex cone $C$, i.e., $\emptyset \neq \text{cor} C \subset C = \mathbb{R}_+ \cdot C = C + C \subset Y$.

**Lemma 2.1.** For every $e \in \text{cor} C$ the following hold:

\[
\text{cor} \quad R_+^* \cdot e - C = Y,
\]

\[
\text{icr} \quad R_+^* \cdot e + C = \text{cor} C.
\]

**Proof.** The inclusion $R_+^* \cdot e - C \subset Y$ is obvious. In order to prove the converse inclusion, consider an arbitrary point $y \in Y$. Since $e \in \text{cor} C$, there exists a real number $\alpha > 0$ such that $e + \alpha(-y) \in C$, which implies that $y \in \frac{1}{\alpha}e - \frac{1}{\alpha}C \subset R_+^* \cdot e - C$. Thus (2) holds.

Since $e \in \text{cor} C$, it follows by Lemma 2.1 in [12] that $\text{cor} C = \text{icr} C = \bigcup_{\alpha > 0} (\alpha e + C) = R_+^* \cdot e + C$, hence (3) is true.
Lemma 2.2. Assume that $C$ is vectorially closed and let $e \in \text{cor} \ C$. Then, for every $x \in Y$, the set

$$A_e(x) := \{ \alpha \in \mathbb{R} \mid x \in \alpha e - C \}$$

is nonempty and closed. Moreover, if $C \neq Y$, then $A_e(x)$ is bounded from below.

Proof. Consider an arbitrary point $x \in Y$. The nonemptiness of $A_e(x)$ follows by (2).

Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence in $A_e(x)$, which converges to some $\alpha \in \mathbb{R}$. Then we have $\alpha_n e - x \in C$ for all $n \in \mathbb{N}$ and $\lim_{n \to +\infty} (\alpha - \alpha_n) = 0$. It follows that $\alpha e - x = (\alpha - \alpha_n)e + (\alpha_n e - x) \in \text{vcl} \ C = C$, hence $\alpha \in A_e(x)$. Thus $A_e(x)$ is a closed set.

Assume now that $C \neq Y$ and suppose to the contrary that $A_e(x)$ is not bounded from below. Then there exists $n_0 \in \mathbb{N}$ such that $-n \in A_e(x)$, i.e., $-e \in \frac{1}{n}x + C$, for all $n \in \mathbb{N} \cap [n_0, +\infty]$. Since $\lim_{n \to +\infty} \frac{1}{n} = 0$ we infer that $-e \in \text{vcl} \ C = C$. Hence we have $0_Y = e + (-e) \in (\text{cor} \ C) + C = \text{cor} \ C$, which yields $C = Y$, a contradiction. Thus $A_e(x)$ is bounded from below. \qed

Remark 2.3. Lemma 2.2 in particular shows that, under the assumption that $C$ is a vectorially closed solid convex cone, $C \neq Y$, one can define for each couple of points $(e, v) \in (\text{cor} \ C) \times Y$ a real-valued function $h_{e,v} : Y \to \mathbb{R}$ by

$$h_{e,v}(y) := \min A_e(y - v), \forall y \in Y.$$ (4)

Remark that, in the particular case when $Y$ is a real topological linear space, partially ordered by a closed convex cone, $C \neq Y$, which has a nonempty interior, the functions $h_{e,v}$ actually are the "smallest strictly monotonic functions" in the sense of Dinh The Luc [7].

3. Explicitly quasiconvex functions

In what follows $S$ will be a nonempty convex subset of a real linear space $X$. For all points $x_1, x_2 \in X$ we denote $][x_1, x_2]: = \{(1 - t)x_1 + tx_2 \mid t \in [0, 1]\}$. Notice that $][x_1, x_2]$ becomes a singleton whenever $x_1 = x_2$.

Definition 3.1. An extended real-valued function $\varphi : S \to \overline{\mathbb{R}}$ is said to be explicitly quasiconvex if for all $x_1, x_2 \in S$ and $x \in ]x_1, x_2[$ one has

$$\varphi(x) \leq \max\{\varphi(x_1), \varphi(x_2)\},$$

where the strict inequality holds whenever $\varphi(x_1) \neq \varphi(x_2)$.

Proposition 3.2. For any function $\varphi : S \to \overline{\mathbb{R}}$ the following assertions are equivalent:
1° \( \varphi \) is explicitly quasiconvex.

2° For all \( \lambda \in \mathbb{R} \cup \{+\infty\} \) and \( x_1, x_2 \in S \) with \( \varphi(x_1) < \lambda \) and \( \varphi(x_2) \leq \lambda \) one has
\( \varphi(x) < \lambda \) for every \( x \in [x_1, x_2[. \)

**Proof.** 1° \( \Rightarrow \) 2°. Assume that 1° holds and let \( \lambda \in \mathbb{R} \cup \{+\infty\} \). Let \( x_1, x_2 \in S \) with \( \varphi(x_1) < \lambda \), \( \varphi(x_2) \leq \lambda \), and let \( x \in [x_1, x_2[. \) If \( \varphi(x_1) = \varphi(x_2) \), then 1° implies \( \varphi(x) \leq \max\{\varphi(x_1), \varphi(x_2)\} = \varphi(x_1) < \lambda \). Otherwise, if \( \varphi(x_1) \neq \varphi(x_2) \), then we can also deduce by 1° that \( \varphi(x) < \max\{\varphi(x_1), \varphi(x_2)\} \leq \lambda \). Hence 2° holds.

2° \( \Rightarrow \) 1°. Assume that 2° holds and consider some arbitrary points \( x_1, x_2 \in S \) and \( x \in [x_1, x_2[. \) Without loss of generality we can assume that \( \varphi(x_1) \leq \varphi(x_2) \).

Case 1: \( \varphi(x_1) = \varphi(x_2) \).

On one hand, if \( \varphi(x_2) = +\infty \), then \( \varphi(x) \leq +\infty = \max\{\varphi(x_1), \varphi(x_2)\} \). On the other hand, if \( \varphi(x_2) \in \mathbb{R} \), then for each real number \( \varepsilon > 0 \), denoting \( \lambda_\varepsilon := \varphi(x_2) + \varepsilon \), we have \( \varphi(x_1) = \varphi(x_2) < \lambda_\varepsilon \). By 2° we can deduce that \( \varphi(x) < \lambda_\varepsilon \) and then, letting \( \varepsilon \searrow 0 \), we get \( \varphi(x) \leq \varphi(x_2) = \max\{\varphi(x_1), \varphi(x_2)\} \).

Case 2: \( \varphi(x_1) \neq \varphi(x_2) \), i.e., \( \varphi(x_1) < \varphi(x_2) \).

In this case, denoting \( \lambda := \varphi(x_2) \), we have \( \lambda \in \mathbb{R} \cup \{+\infty\} \), \( \varphi(x_1) < \lambda \), and \( \varphi(x_2) \leq \lambda \). By 2° it follows that \( \varphi(x) < \lambda = \max\{\varphi(x_1), \varphi(x_2)\} \).

Let \( Y \) be a linear space over the field \( \mathbb{R} \) of real numbers, partially ordered by a solid convex cone \( C \), i.e., \( C = C + C \subset Y \) and
\[
\text{cor } C := \{ x \in C \mid \forall y \in Y, \exists \lambda \in \mathbb{R}^*_+, x + [0, \lambda] \cdot y \subset C \} \neq \emptyset.
\]

The following generalized convexity concept has been recently introduced by Popovici in [12]:

**Definition 3.3.** A set-valued map \( F : S \to 2^Y \), defined on a nonempty convex subset \( S \) of \( X \), is said to be explicitly \( C \)-quasiconvex if for all \( x_1, x_2 \in S \) and \( x \in [x_1, x_2[ \) the following inclusion holds:
\[
(F(x_1) + \text{cor } C) \cap (F(x_2) + C) \subset F(x) + \text{cor } C.
\]

A function \( f : S \to Y \) is called explicitly \( C \)-quasiconvex if the set-valued map \( F : S \to 2^Y \), defined for all \( x \in S \) by \( F(x) = \{ f(x) \} \), is explicitly \( C \)-quasiconvex.

**Proposition 3.4.** Let \( \Phi : S \to 2^\mathbb{R} \) be a set-valued map and let \( \varphi : S \to \mathbb{R} \) be its marginal function, defined for all \( x \in S \) by \( \varphi(x) = \inf \Phi(x) \). Then the following hold:

a) If \( \varphi \) is explicitly quasiconvex, then \( \Phi \) is explicitly \( \mathbb{R}_+ \)-quasiconvex.
b) If \( \Phi \) is explicitly \( \mathbb{R}_+ \)-quasiconvex and its values are nonempty and closed, then \( \varphi \) is explicitly quasiconvex.

**Proof.** a) Assuming that \( \varphi \) is explicitly quasiconvex, consider some arbitrary \( x_1, x_2 \in S \) and \( x \in [x_1, x_2[. \) Then, for any \( \lambda \in (\Phi(x_1) + \text{int}\mathbb{R}_+) \cap (\Phi(x_2) + \mathbb{R}_+) \), there exist \( y_1 \in \Phi(x_1) \) and \( y_2 \in \Phi(x_2) \) such that \( \varphi(x_1) \leq y_1 < \lambda \) and \( \varphi(x_2) \leq y_2 \leq \lambda \). Since \( \varphi \) is explicitly quasiconvex, it follows by Proposition 3.2 that \( \varphi(x) < \lambda \), i.e., \( \inf \Phi(x) < \lambda \), which shows that \( \lambda \in \Phi(x) + \text{int}\mathbb{R}_+ \). We infer that \( (\Phi(x_1) + \text{int}\mathbb{R}_+) \cap (\Phi(x_2) + \mathbb{R}_+) \subset \Phi(x) + \text{int}\mathbb{R}_+ \), proving that \( \Phi \) is explicitly \( \mathbb{R}_+ \)-quasiconvex.

b) Suppose that \( \Phi \) is explicitly \( \mathbb{R}_+ \)-quasiconvex. In order to prove that \( \varphi \) is explicitly quasiconvex we will use Proposition 3.2. Consider some arbitrary \( \lambda \in \mathbb{R} \cup \{+\infty\}, x_1, x_2 \in S \) with \( \varphi(x_1) < \lambda \) and \( \varphi(x_2) \leq \lambda \), and \( x \in [x_1, x_2[. \) We have to prove that

(5) \( \varphi(x) < \lambda. \)

**Case 1:** \( \lambda = +\infty. \)

Since \( \Phi(x) \) is not empty, we have \( \varphi(x) \in \{-\infty\} \cup \mathbb{R} \), hence (5) obviously holds.

**Case 2:** \( \lambda \in \mathbb{R}. \)

On one hand, since \( \inf \Phi(x_1) = \varphi(x_1) < \lambda \) we have \( \lambda \in \Phi(x_1) + \text{int}\mathbb{R}_+. \) On the other hand, since \( \inf \Phi(x_2) = \varphi(x_2) \leq \lambda \) and \( \Phi(x_2) \) is closed, we can deduce that \( \lambda \in \Phi(x_2) + \mathbb{R}_+. \) By explicit \( \mathbb{R}_+ \)-quasiconvexity of \( \Phi \) it follows that \( \lambda \in (\Phi(x_1) + \text{int}\mathbb{R}_+) \cap (\Phi(x_2) + \mathbb{R}_+) \subset \Phi(x) + \text{int}\mathbb{R}_+ \), which yields the existence of some \( y_\varepsilon \in \Phi(x) \) with \( \varphi(x) \leq y_\varepsilon < \lambda. \) Thus (5) holds.

**Remark 3.5.** As shown by the following example, the closeness assumption imposed on \( \Phi \) is essential in Proposition 3.4 (b).

**Example 3.6.** Let \( \Phi : \mathbb{R} \to 2^\mathbb{R} \) be the set-valued map defined by

\[
\Phi(x) := \begin{cases} 
|1, +\infty[ & \text{if } x \neq 0 \\
|0, +\infty[ & \text{if } x = 0.
\end{cases}
\]

It is easily seen that for all \( x_1, x_2 \in \mathbb{R} \) and \( \lambda \in (\Phi(x_1) + \text{int}\mathbb{R}_+) \cap (\Phi(x_2) + \mathbb{R}_+) \), we actually have \( \lambda > 1 \), hence \( \lambda \in \Phi(x) + \text{int}\mathbb{R}_+ \) for all \( x \in [x_1, x_2[. \) Thus \( \Phi \) is explicitly \( \mathbb{R}_+ \)-quasiconvex. However, its marginal function, given by

\[
\varphi(x) := \inf \Phi(x) = \begin{cases} 
1 & \text{if } x \neq 0 \\
0 & \text{if } x = 0,
\end{cases}
\]

is not explicitly quasiconvex.
Theorem 3.7. Consider an arbitrary point $e \in \text{cor } C$ and let $F : S \to 2^Y$ be a set-valued map such that $F(x) + C$ is nonempty and vectorially closed for all $x \in S$. For any $v \in Y$, define the set-valued map $\Phi_v : S \to 2^Y$ by

$$\Phi_v(x) := \{ \alpha \in \mathbb{R} \mid v + \alpha e \in F(x) + C \} \text{ for all } x \in S,$$

and denote by $\varphi_v : S \to \mathbb{R}$ its marginal function, i.e.,

$$\varphi_v(x) = \inf \Phi_v(x) \text{ for all } x \in S.$$

The following assertions are equivalent:

1° The map $F$ is explicitly $C$-quasiconvex.

2° For every $v \in Y$ the map $\Phi_v$ is explicitly $\mathbb{R}_+$-quasiconvex.

3° For every $v \in Y$ the function $\varphi_v$ is explicitly quasiconvex.

Proof. In order to prove the equivalence $2° \Leftrightarrow 3°$ we will use Proposition 3.4 for $\Phi_v$ in the role of $\Phi$, where $v$ stands for some arbitrary point in $Y$. We just have to show that for any $x \in S$ the set $\Phi_v(x)$ is nonempty and closed. Indeed, $F(x) + C$ being nonempty, we can choose a point $y \in F(x)$. Since $e \in \text{cor } C$, it follows by (2) that $y - v \in \mathbb{R}_+ \cdot e - C$, hence $\emptyset \neq \{ \alpha \in \mathbb{R} \mid y - v \in \alpha e - C \} = \{ \alpha \in \mathbb{R} \mid y \in v + \alpha e - C \} \subset \Phi_v(x)$. Thus $\Phi_v(x)$ is nonempty. Consider now a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of numbers in $\Phi_v(x)$, which converges to some $\alpha \in \mathbb{R}$, i.e., $v + \alpha_n e \in F(x) + C$ for all $n \in \mathbb{N}$ and $\lim_{n \to +\infty} (\alpha - \alpha_n) = 0$. Since $v + \alpha e = v + \alpha_n e + (\alpha - \alpha_n)e$ for all $n \in \mathbb{N}$, it follows that $v + \alpha e \in \text{vcl}(F(x) + C) = F(x) + C$, hence $\alpha \in \Phi_v(x)$. Thus $\Phi_v(x)$ is closed.

Let us now prove the implication $1° \Rightarrow 2°$. Under the assumption that $F$ is explicitly $C$-quasiconvex, consider an arbitrary point $v \in Y$. In order to prove that $\Phi_v$ is explicitly $\mathbb{R}_+$-quasiconvex, let $x_1, x_2 \in S$, let $x \in [x_1, x_2]$, and let $\lambda \in (\Phi_v(x_1) + \text{int } \mathbb{R}_+) \cap (\Phi_v(x_2) + \mathbb{R}_+)$. Then there exist $\alpha_1 \in \Phi_v(x_1)$ and $\alpha_2 \in \Phi_v(x_2)$ such that $\alpha_1 < \lambda$ and $\alpha_2 \leq \lambda$. On one hand we have $v + \lambda e = v + \alpha_1 e + (\lambda - \alpha_1) e \in F(x_1) + C + \text{cor } C$. On the other hand we also have $v + \lambda e = v + \alpha_2 e + (\lambda - \alpha_2) e \in F(x_2) + C + C \subset F(x_2) + C$. By explicit $C$-quasiconvexity of $F$ we deduce that

$$v + \lambda e \in (F(x_1) + C + \text{cor } C) \cap (F(x_2) + C) \subset F(x) + \text{cor } C.$$

By (3) we infer the existence of a positive number $\tau > 0$ such that $v + \lambda e \in \tau e + F(x) + C$, i.e., $v + (\lambda - \tau) e \in F(x) + C$, which actually means that $\lambda - \tau \in \Phi_v(x)$. It follows that $\lambda \in \Phi_v(x) + \tau \subset \Phi_v(x) + \text{int } \mathbb{R}_+$. Thus we have $(\Phi_v(x_1) + \text{int } \mathbb{R}_+) \cap (\Phi_v(x_2) + \mathbb{R}_+) \subset \Phi_v(x) + \text{int } \mathbb{R}_+$.

It remains to prove the implication $2° \Rightarrow 1°$. To this end, assume that $2°$ holds and suppose to the contrary that $1°$ is not true. Then there exist $x_1^0, x_2^0 \in S$, $x^0 \in [x_1^0, x_2^0]$, and let $y \in F(x^0) + C$.
and $y^0 \in (F(x_1^0) + \text{cor} C) \cap (F(x_2^0) + C) \setminus (F(x_0^0) + \text{cor} C)$. On one hand, since $y^0 \in F(x_1^0) + \text{cor} C$, we can find, by virtue of (3), a number $\tau_0 > 0$ such that $y_0 = \tau_0 e + F(x_1^0) + C$, i.e., $y_0 + (-\tau_0)e \in F(x_1^0) + C$. This means that $-\tau_0 \in \Phi_{\rho}(x_1^0)$, hence $0 \in \Phi_{\rho}(x_1^0) + \tau_0 \subset \Phi_{\rho}(x_1^0) + \text{int}\mathbb{R}_+$. On the other hand, since $y^0 + 0 \cdot e = y^0 \in F(x_2^0) + C$, we also have $0 \in \Phi_{\rho}(x_2^0) \subset \Phi_{\rho}(x_2^0) + \mathbb{R}_+$. Hence $0 \in (\Phi_{\rho}(x_1^0) + \text{int}\mathbb{R}_+) \cap (\Phi_{\rho}(x_2^0) + \mathbb{R}_+)$. Taking into account that $\Phi_{\rho}$ is explicitly $\mathbb{R}_+$-quasiconvex (by 2°), we infer that $0 \in \Phi_{\rho}(x_0^0) + \text{int}\mathbb{R}_+$. Thus there exists $\beta > 0$ such that $-\beta \in \Phi_{\rho}(x_0^0)$, i.e., $y^0 - \beta e \in F(x_0^0) + C$. It follows that $y^0 \in F(x_0^0) + C + \beta e \subset F(x_0^0) + C + \mathbb{R}_+^\ast \cdot \text{cor} C = F(x_0^0) + \text{cor} C$, which contradicts the fact that $y^0 \notin F(x_0^0) + \text{cor} C$. □

**Corollary 3.8.** Assume that $C \neq Y$ is vectorially closed and consider an arbitrary point $e \in \text{cor} C$. For any vector-valued function $f : S \to Y$ the following assertions are equivalent:

1° $f$ is explicitly $C$-quasiconvex.

2° The composite function $h_{e \cdot v} \circ f : S \to \mathbb{R}$ is explicitly quasiconvex, for every $v \in Y$.

*Proof.* The desired equivalence directly follows from Theorem 3.7, applied to the single-valued map defined as $F(x) := \{f(x)\}$ for all $x \in S$. Indeed, since $C$ is vectorially closed, the set $F(x) + C = f(x) + C$ is nonempty and vectorially closed, for all $x \in S$. Moreover, for all $v \in Y$ and $x \in S$, we have $\varphi_v(x) := \inf \Phi_v(x) = \inf\{\alpha \in \mathbb{R} \mid v + \alpha e \in f(x) + C\}$. In view of Remark 4, this means that $\varphi_v(x) = h_{e \cdot v}(f(x))$ for all $x \in S$, i.e., $\varphi_v = h_{e \cdot v} \circ f$. □

**Remark 3.9.** The well-known characterization of cone-quasiconvex vector-valued functions obtained by Dinh The Luc in [7] (Proposition 1.6.3), as well as the characterization of cone-quasiconvex set-valued maps established by Benoist and Popovici in [4] (Theorem 3.2) under the framework of a real topological vector space, partially ordered by a closed convex cone with nonempty interior, can be extended in the general setting of our paper. Actually, by replacing the linear segments by continuous arcs as in [6], farther generalizations of these characterization theorems may be easily derived from our results.

**References**


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