ON INVERSE PROBLEMS FOR ITERATED FUNCTION SYSTEMS

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Abstract

In this paper we recall inverse problems for iterated function systems and how these problems can be reduced to optimization problems. We extend these results to IFS operators with place dependent probabilities.

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1 Introduction

The basic idea of Iterated Function Systems (IFS) can be traced back to some historical papers but the use of such systems to construct fractals and other similar sets was first described by Hutchinson (1981) and Barnsley (1985). Some possible applications of IFS can be found in image processing theory, signal analysis, fractal approximations, stochastic growth models, ect. The fundamental result on which the IFS method is based is Banach theorem. In practical applications, the following inverse problem is fundamental: given f in a complete metric space (X,d), find a contractive operator T that admits a unique fixed point f* such that d(f, f*) is small enough. In fact if one is able to solve the inverse problem with arbitrary precision, it is possible to identify f with the operator T which has it as fixed point. In particular in the present work, we analyze some contractive operators on C([a, b]) which arise as generalizations of adjoint operators associated to invariant measures of IFS and we show how the inverse problems can be formulated as constrained convex optimization problems. We then extend these to the case of place dependent probabilities.

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2 Contractive maps on $C([a, b])$

The aim of this section is to recall some basic facts on IFS and how a class of contractions on $C([a, b])$ arise as adjoint operators of iterated function systems. If we consider the space $H([a, b])$, built with all compact subsets of $[a, b]$, and if we introduce on $H([a, b])$ the following metric (Hausdorff metric):

$$h(A, B) = \max \{ \max_{x \in A} \min_{y \in B} |x - y|, \min_{x \in B} \max_{y \in A} |x - y| \}$$

then the space $(H([a, b]), h)$ is a complete metric space ([7]). Now suppose that $\mathcal{B}([a, b])$ be the $\sigma$-algebra of Borel subsets of $[a, b]$ and $M([a, b])$ denote the space of all probability measures on $[a, b]$. It is possible to define a metric on $M([a, b])$ as:

$$d_H(\mu, \nu) = \sup_{f \in L^1} \int_a^b f \, d\mu - \int_a^b f \, d\nu, \quad \mu, \nu \in M([a, b])$$

where $L^1 = \{ f : [a, b] \to \mathbb{R} : |f(x) - f(y)| \leq |x - y| \}$ and $(M([a, b]), d_H)$ is a complete metric space [7] (this metric is called Hutchinson metric). We now build a classical IFS operator; to do this, let $w = \{ w_1, w_2, \ldots, w_n \}$ be a set of $n$ continuous contraction maps on $[a, b]$, i.e. $w_i : [a, b] \to [a, b]$ and:

$$|w_i(x) - w_i(y)| \leq c_i |x - y|, \quad x, y \in [a, b], \quad 0 \leq c_i < 1, \quad i = 1 \ldots n.$$ 

The couple $([a, b], \mu)$ is called Iterated Function Systems (IFS). In many cases it is convenient to define the maximum contractivity factor of the IFS as:

$$c = \max_{i=1}^n c_i < 1.$$ 

Associated with these maps there is a set of non-zero probabilities, $p = \{ p_1, p_2, \ldots, p_n \}$, $p_i > 0$ and $\sum_{i=1}^n p_i = 1$. Now for a set $S \in H([a, b])$, denote $w_i(S) = \{ w_i(x), x \in S \}$ and the set valued map $w$ defined as:

$$w(S) = \bigcup_{i=1}^n w_i(S).$$

Also define the iteration sequence $w^{s+1}(S) = w(w^s(S)), s = 1, 2 \ldots$ Two important results for contractive IFS are given below.

**Theorem 2.1.** [6]

i) There exists a unique compact subset $A \in H([a, b])$, the attractor of the IFS $\{ [a, b], w \}$ (independent of $p$) such that:

$$A = w(A) = \bigcup_{i=1}^n w_i(A)$$

and $h(w^s(S), A) \to 0$ as $s \to \infty$ for all $S \in H([a, b])$. 
ii) Define the following "Markov operator" $M : M([a, b]) \to M([a, b])$,

$$M(\nu) = \sum_{i=1}^{n} p_i \nu \circ w_i^{-1}.$$ 

Then there exists a unique measure $\mu \in M([a, b])$, termed the invariant measure, which obeys the fixed point condition:

$$M\mu = \mu.$$ 

Moreover, $\text{supp}(\mu) = A$.

We now introduce contractive maps on $C([a, b])$ associated with classical IFS operators. Then for a $\mu$-integrable function $f : [a, b] \to \mathbb{R}$,

$$\int_{A} f(x) d\mu(x) = \sum_{i=1}^{n} p_i \int_{A} f(w_i(x)) d\mu(x).$$

Let $C([a, b])$ the Banach space of continuous functions on $[a, b]$; associated with the IFS $\{w, p\}$ define the following operator $T : C([a, b]) \to C([a, b])$:

$$Tf = \sum_{i=1}^{n} p_i f \circ w_i, \ f \in C([a, b]).$$

Now for a given $\mu \in M([a, b])$, define the linear functional $F : C([a, b]) \to \mathbb{R}$,

$$F(f) = \langle f, \nu \rangle = \int_{[a,b]} f d\nu.$$ 

Then:

$$\langle Tf, \nu \rangle = \langle f, M\nu \rangle$$

i.e. $T$ is the adjoint operator of $M$.

The operator $T$ is a contraction on the complete metric space $(C([a, b]), d_\infty)$ where

$$d_\infty(f, g) = \max_{x \in [a, b]} |f(x) - g(x)|$$

with contractivity factor $c = \max_{i=1,...,n} p_i < 1$ [6]. Now let $f \in C([a, b])$; then the inverse problem consists of finding an operator $T : C([a, b]) \to C([a, b])$, $Tf = \sum_{i=1}^{n} p_i f \circ w_i$, such that $Tf = f$. In approximating way this problem can be solved using the following result (known as Collage theorem).

**Theorem 2.2.** [6] Let $f \in C([a, b])$. If $d(Tf, f) \leq \epsilon$ then $d(f, \tilde{f}) \leq \frac{\epsilon}{1-c}$, where $T\tilde{f} = \tilde{f}$ and $c = \max_{i=1,...,n} p_i < 1$.

So the inverse problem is reduced to the problem of minimizing the quantity $d(Tf, f)$ over a functional space built with the maps $w_i$ and the probabilities $p_i$. So if a family of maps $w_i$ with associated probabilities $p_i$ is given, as in [6], the inverse problem can be reduced to a convex constrained optimization problem on $\mathbb{R}^n$. The aim of the next sections is to introduce some generalizations of the map $T$ and to study the inverse problems.
3 A generalized contractive operator of IFS type on \( C([a, b]) \)

We now recall some results proved in [9]. We consider the function \( T_1 : C([a, b]) \to C([a, b]) \), which is a generalization of the operator \( T \), defined as:

\[
T_1 u(x) = \sum_{i=1}^{n} p_i \phi_i \circ u \circ w_i(x)
\]

where \( \phi_i : \mathbb{R} \to \mathbb{R} \) are Lipschitzian with Lipschitz constants \( K_i \in \mathbb{R}_+ \), \( w_i : [a, b] \to [a, b] \) are continuous and \( p_i \in \mathbb{R}_+ \), \( i = 1 \ldots n \). In [9] we proved the following result which establishes that, under some hypotheses, \( T_1 \) is a contraction on \( C([a, b]) \). We observe that \( T_1 \) is reduced to \( T \) when \( \phi_i(x) = x \), \( \forall i = 1 \ldots n \).

**Theorem 3.1.** [9] \( T_1 : C([a, b]) \to C([a, b]) \) verifies the following inequality:

\[
d_\infty(T_1 f, T_1 g) \leq \left\{ \sum_{i=1}^{n} p_i K_i \right\} d_\infty(f, g).
\]

Banach theorem ensures that if \( K_{T_1} := \sum_{i=1}^{n} p_i K_i \) is less than one then the map \( T_1 \) is a contraction on \( C([a, b]) \); so the fixed point equation \( T_1 f = f \) has a unique solution \( \tilde{f} \) and the sequence \( T_1^n u = T_1(T_1^{n-1} u) \) converges to \( \tilde{f} \) for all \( u \in C([a, b]) \).

Given a function \( f \in C([a, b]) \) the inverse problem consists of finding a contractive operator \( T_1 \), as above, such that the fixed point of \( T_1 \) is near to \( f \). If the Lipschitzian maps \( \phi_i \) and the continuous functions \( w_i \) are given, the solutions of the inverse problem have to be found by using the parameters \( p_i \) (we will write \( T_{1,p} \) instead of \( T_1 \) to put in evidence the dependence of the operator \( T_1 \) on the vector \( p = (p_1, p_2, \ldots, p_n) \)). Furthermore, since the fixed point \( \tilde{f}_p \) of a map \( T_{1,p} \) is unknown, is useful the following theorem which gives us an estimate of \( \tilde{f}_p \).

**Theorem 3.2.** [9] Let \( p^* \in \mathbb{R}_+^n \) such that \( K_{T_{1,p}} = \sum_{i=1}^{n} p_i^* K_i < 1 \) and \( f \in C([a, b]) \). If \( \tilde{f}_{p^*} \) is the unique fixed point of \( T_{1,p^*} \) and if \( d_\infty(f, T_{1,p^*} f) < \epsilon \) then:

\[
d_\infty(f, \tilde{f}_{p^*}) < \frac{\epsilon}{1 - K_{T_{1,p}}}
\]

In other words, the previous theorem states that if \( f \) is given and \( p^* \) is such that \( d_\infty(T_{1,p^*} f, f) < \epsilon \) (with \( \epsilon \) "sufficiently small") then, \( \forall u \in C([a, b]) \), the sequence \( T_{1,p^*}^s u = T_{1,p^*}(T_{1,p^*}^{s-1} u) \) converges to \( \tilde{f}_{p^*} \) (when \( s \to +\infty \)) which is an approximation of \( f \). Obviously now the problem consists of finding \( p^* \in \mathbb{R}_+^n \) such that:

\[ P1 \quad F(p^*) = \min F(p) := \min d_\infty(T_{1,p} f, f) \]

with \( \sum_{i=1}^{n} p_i K_i \leq \delta \) and \( \delta < 1 \). We will see that this is a constrained convex optimization problem. It is clear that the ideal solution consists of finding \( p^* \) such that \( F(p^*) = 0 \); in fact in this case the map \( T_{1,p} \) has exactly \( f \) as fixed point. Since \( F \) is in general not differentiable, this is a nonsmooth equation. Furthermore \( F \) is
convex and then semismooth; this assures the convergence of the Newton’s method introduced by Qi and Sun [11]. In the other cases the value $F(p^*)$ is a lower bound for the precision of the estimate; in this setting this bound can be improved only increasing the number $n$ of parameters $p_i$.

With the following results we study the previous minimization problem and we give a necessary and sufficient condition for the existence of a minimizer.

**Theorem 3.3.** [9] $F : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is convex.

**Proposition 3.1.** [9] The set $C = \{p \in \mathbb{R}_+^n : \sum_{i=1}^n p_i K_i \leq \delta\}$ is convex.

So the problem $P_1$ is a constrained convex optimization problem. A necessary and sufficient condition for the existence of minimizer, given by subdifferential theory ([12]), is given in the following result.

**Theorem 3.4.** [9] $p^* \in C$ is a minimizer of $F(p)$ if and only if there exist $\lambda_i \geq 0$, $i = 1 \ldots n + 1$ such that $p^*$ is a solution of the following system:

$$
\begin{align*}
\lambda_i &\geq 0, \ i = 1 \ldots n + 1 \\
\sum_{i=1}^n p_i K_i - \delta &\leq 0, \\
\lambda_i p_i &\leq 0, -p_i \leq 0, \ i = 1 \ldots n \\
\lambda_{n+1}(\sum_{i=1}^n p_i K_i - \delta) &\leq 0
\end{align*}
$$

4 IFS operators with place dependent probabilities

In this section we introduce a generalization of the previous operator in the case of place dependent probabilities; we consider on $(\mathcal{C}([a,b]), d_\infty)$ the following map

$$T_2 u(x) = \sum_{i=1}^n p_i(x) \phi_i \circ u \circ w_i(x)$$

where $w_i$ are defined as above and $p_i : [a,b] \rightarrow [0,1]$ are continuous functions such that $0 \leq \sum_{i=1}^n p_i(x) < 1$ for all $i = 1 \ldots n$. Under these assumptions it is clear that $T_2 : \mathcal{C}([a,b]) \rightarrow \mathcal{C}([a,b])$. The following result states that $T_2$ is a contraction on $\mathcal{C}([a,b])$.

**Theorem 4.1.** $T_2$ is a contraction on $\mathcal{C}([a,b])$.

*Proof.* In fact we have:

$$d_\infty(Tf, Tg) = \sup_{x \in [a,b]} |Tf(x) - Tg(x)| \leq$$

$$\sup_{x \in [a,b]} \sum_{i=1}^n p_i(x) |f(w_i(x)) - g(w_i(x))| \leq \sum_{i=1}^n p_i(x) d_\infty(f, g) \leq Md_\infty(f, g)$$

$\square$
So the functional equation $T_2 f(x) = f(x)$ has a unique solution $f^*$ in $C([a, b])$. Suppose now that we have two sets of probabilities $p_i(x)$ and $p_i^*(x)$ such that $\sum_{i=1}^{n} p_i(x) < 1$ and $\sum_{i=1}^{n} p_i^*(x) < 1$. Let $f$ and $f^*$ be two fixed points on the operator $T_2$ and $T_2^*$. The following result states a stability result.

**Theorem 4.2.** $d_{\infty}(f, f^*) \leq \frac{M_1 \sum_{i=1}^{n} |p_i(x) - p_i^*(x)|}{1 - M_2}$.

**Proof.** We have:

$$d_{\infty}(f, f^*) = d_{\infty}(T_2 f, T_2^* f^*) = \sup_{x \in [a, b]} |T_2 f(x) - T_2^* f^*(x)| \leq$$

$$\sup_{x \in [a, b]} \left| \sum_{i=1}^{n} p_i(x) f(x) - \sum_{i=1}^{n} p_i^* f(x) \right| + \sup_{x \in [a, b]} \left| \sum_{i=1}^{n} p_i^*(x) f(x) - \sum_{i=1}^{n} p_i^* f^*(x) \right| \leq$$

$$M \sum_{i=1}^{n} |p_i(x) - p_i^*(x)| + d_{\infty}(T_2^* f, T_2^* f^*) \leq M_1 \sum_{i=1}^{n} |p_i(x) - p_i^*(x)| + M_2 d_{\infty}(f, f^*) .$$

Then

$$d_{\infty}(f, f^*) \leq \frac{M_1 \sum_{i=1}^{n} |p_i(x) - p_i^*(x)|}{1 - M_2}$$

\[ \square \]

**References**


